

IRREGULAR RIEMANN-HILBERT CORRESPONDENCE, ALEKSEEV-MEINRENKEN DYNAMICAL r -MATRICES AND DRINFELD TWISTS

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Abstract

In 2004, Enriquez-Etingof-Marshall suggested a new approach to the Ginzburg-Weinstein linearization theorem. This approach is based on solving a system of PDEs for a gauge transformation between the standard classical r -matrix and the Alekseev-Meinrenken dynamical r -matrix. In this paper, we explain that this gauge transformation can be constructed as a monodromy (connection matrix) for a certain irregular Riemann-Hilbert problem. This further indicates a surprising relation between the connection matrix and Drinfeld twist. Our construction is based on earlier works by Boalch. As byproducts, we get a symplectic neighborhood version of the Ginzburg-Weinstein linearization theorem as well as a new description of the Lu-Weinstein symplectic double.

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1 Introduction and main results

Dynamical r -matrices and Stokes phenomenon

In the study of non-commutative Weil algebra [4], Alekseev and Meinrenken introduced a particular dynamical r -matrix r_{AM} , which is an important special case of classical dynamical r -matrices ([24], [22]). Let \mathfrak{g} be a complex reductive Lie algebra and $t \in S^2(\mathfrak{g})^2$ the element corresponding to an invariant inner product on \mathfrak{g} , then r_{AM} , as a map from \mathfrak{g}^* to $\mathfrak{g} \wedge \mathfrak{g}$, is defined by

$$r_{AM}(x) := (\text{id} \otimes \phi(\text{ad}_x^\vee))(t), \quad \forall x \in \mathfrak{g}^*,$$

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where $x^\vee = (x \otimes \text{id})(t)$ and $\phi(z) := -\frac{1}{z} + \frac{1}{2}\coth\frac{z}{2}$, $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}^*$. Remarkably, this r -matrix came to light naturally in two different applications, i.e., in the context of equivariant cohomology [4] and in the description of a Poisson structure on the chiral WZNW phase space compatible with classical G -symmetry [7].

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a classical r -matrix such that $r + r^{2,1} = t$ (thus (\mathfrak{g}, r) is a quasitriangular Lie bialgebra). In [18], Enriquez, Etingof and Marshall constructed formal Poisson isomorphisms between the formal Poisson manifolds \mathfrak{g}^* and G^* (the dual Poisson Lie group). Here \mathfrak{g}^* is equipped with its Kostant-Kirillov-Souriau structure, and G^* with its Poisson-Lie structure given by r . Their result relies on constructing a formal map $g : \mathfrak{g}^* \rightarrow G$ satisfying the following gauge transformation equation (as identity of formal maps $\mathfrak{g}^* \rightarrow \wedge^2(\mathfrak{g})$)

$$g_1^{-1}d_2(g_1) - g_2^{-1}d_1(g_2) + (\otimes^2 \text{Ad}_g)^{-1}r_0 + \langle \text{id} \otimes \text{id} \otimes x, [g_1^{-1}d_3(g_1), g_2^{-1}d_3(g_2)] \rangle = r_{\text{AM}}, \quad (1)$$

Here $r_0 := \frac{1}{2}(r - r^{2,1})$, $g_1^{-1}d_2(g)(x) = \sum_i g^{-1}\partial_{\varepsilon^i}g(x) \otimes e_i$ is viewed as a formal function $\mathfrak{g}^* \rightarrow \mathfrak{g}^{\otimes 2}$, $\{e_i\}$ is a basis of \mathfrak{g} , $\{\xi^i\}$ the corresponding coordinates on \mathfrak{g}^* and $g_i^{-1}d_j(g_i) = (g_1^{-1}d_2(g_1))^{i,j}$. Two constructions of solutions of (1) are given: the first one uses the theory of the classical Yang-Baxter equation and gauge transformations; the second one relies on the theory of quantization of Lie bialgebras. The result in [18] may be viewed as a generalization of the formal version of [25], in which Ginzburg and Weinstein proved the existence of a Poisson diffeomorphism between the real Poisson manifolds k^* and K^* , where K is a compact Lie group and k is its Lie algebra. Different approaches to similar results in the subject of linearization of Poisson structures can be found in [1] and [9].

The main purpose of the present paper is to give an explicit solution of the above equation (provided r is a standard classical r -matrix). This allows us to understand the geometric meaning of equation (1) and clarify its relation with irregular Riemann-Hilbert correspondence. The solutions will be constructed as the monodromy of certain differential equations with irregular types. To be precise, let us consider the meromorphic connection on the trivial holomorphic principal G -bundle P on \mathbb{P}^1 which has the form

$$\nabla = d - \left(\frac{A_0}{z^2} + \frac{x}{z} \right) dz \quad (2)$$

where $A_0, x \in \mathfrak{g}$. We assume that $A_0 \in \mathfrak{t}_{\text{reg}}$ and once fixed, the only variable is $x \in \mathfrak{g} \cong \mathfrak{g}^*$ (via the inner product on \mathfrak{g}). Then we consider the monodromy of ∇ from 0 to ∞ , known as the connection matrix $C(x)$ of ∇ , which is computed as the ratio of two canonical solutions of $\nabla F = 0$, one is around ∞ and another is on one chosen Stokes sector at 0. Thus we get a map $C : \mathfrak{g}^* \rightarrow G$ by mapping $x \in \mathfrak{g}^*$ to the connection matrix $C(x)$ of ∇ . See Section 3 for more details. One of the main results of this paper is

Theorem 1.1. *The map $C_{2\pi i} \in \text{Map}(\mathfrak{g}^*, G)$, defined by $C_{2\pi i}(x) := C(\frac{1}{2\pi i}x)$ for all $x \in \mathfrak{g}^*$, is a solution of equation (1).*

The meromorphic connections ∇ taking the form of (2) were previously studied by Boalch. In particular, link between the connections ∇ and the dual Poisson Lie group G^* was discovered in [9], where the Poisson manifold G^* is proven to be a space of Stokes data, and local analytic isomorphisms \mathfrak{g}^* to G^* in a neighbourhood of 0 were constructed. Furthermore, the connection matrix C was used by Boalch to construct the Duistermaat twist [16].

Connection matrices and Drinfeld twists

Having proved the connection matrix satisfies the gauge transformation equation, we can further discuss its relation with Drinfeld twist. This is based on a series work of Enriquez, Etingof and others. In [18], the gauge transformation equation (1) was interpreted as the classical limit of a vertex-IRF transformation equation (see [19]) between a dynamical twist $J_d(x) \in \text{Map}(\mathfrak{g}^*, U(\mathfrak{g})^{\otimes 2}[[\hbar]])$ and a constant twist $J_c \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$. Here $J_d(x)$ and J_c are respectively the twist quantization of r_{AM} and r_0 associated to an admissible associator Φ . As a result, the classical limit of a vertex-IRF transformation $\rho \in \text{Map}(\mathfrak{g}^*, U(\mathfrak{g})[[\hbar]])$ gives rise to a solution of (1). According to [17][18], an admissible Drinfeld twist $J \in U(\mathfrak{g})^{\otimes 2}$ (killing the associator Φ) produces such a vertex-IRF transformation. Here J kills Φ means that J satisfies the identity

$$\Phi = (J^{2,3} J^{1,23})^{-1} J^{1,2} J^{12,3}. \quad (3)$$

Thus in particular, the classical limit of an admissible Drinfeld twist J provides a solution of (1). Now we have two sources of solutions (as is shown before, another one is from connection matrix C), whose relation can be encoded in the following theorem. See section 4 for more details.

Theorem 1.2. *For each (rescaled) connection matrix $C_{2\pi i} \in \text{Map}(\mathfrak{g}^*, G)$, there exists a Drinfeld twist killing the associator Φ whose classical limit is $C_{2\pi i}$.*

In particular, let Φ be the Knizhnik-Zamolodchikov (KZ) associator Φ_{KZ} , which is constructed as the monodromy from 1 to ∞ of the KZ equation on \mathbb{P}^1 with three simple poles at 0, 1, ∞ . Naively the confluence of two simple poles at 0 and 1 in the KZ equation turns the monodromy representing KZ associator to the monodromy C_{\hbar} representing the connection matrix of certain differential equation with one degree two pole. Recall that the connection matrix C is the monodromy from 0 to ∞ of the equation $\nabla F = 0$. Then the above theorem indicates that the monodromy C_{\hbar} is related to certain Drinfeld twist killing Φ_{KZ} . In other words, the confluence of two simple poles in KZ equation may be related to the Drinfeld twist identity (3). The precise relation between Stokes phenomenon and the theory of quantum groups is worked out recently by Toledano Laredo in [33] and in our joint work [34].

Irregular Riemann-Hilbert correspondence

In the second part of this paper, we clarify the relation between the gauge transformation equation (1) and certain irregular Riemann-Hilbert correspondence. This is motivated and based on Boalch's works, e.g. [10] [11] [12] [13], on the study of the geometry of moduli spaces of meromorphic connections on a trivial holomorphic principal G (a complex reductive Lie group)-bundle on Riemann surfaces with divisors. We next present a brief review of these works. In [10], natural symplectic structures were found and described on such moduli spaces both explicitly and from an infinite dimensional viewpoint (generalising the Atiyah-Bott approach). Explicitly, the extended moduli space (see Definition 2.6 of [10]) of meromorphic connections on a trivial G -bundle P over \mathbb{P}^1 , with poles on an effective divisor $D = \sum_{i=1}^m k_i(a_i)$ and a fixed irregular type at each a_i , was proven to be isomorphic to the symplectic quotient of the form $\tilde{O}_1 \times \cdots \times \tilde{O}_m // G$, where \tilde{O}_i is an extended orbit with natural symplectic structure associated to the irregular type at a_i . In [12], a family of new examples of complex quasi-Hamiltonian G -spaces $\tilde{\mathcal{C}}$ with G -valued moment maps was introduced, as generalization of the conjugacy class example of Alekseev-Malkin-Meinrenken [3]. It was further shown that given the divisor $D = \sum_{i=1}^m k_i(a_i)$, the symplectic spaces of monodromy data for meromorphic connections on P with poles on D and fixed irregular types is isomorphic

to the quasi-Hamiltonian quotient space $\tilde{\mathcal{C}}_1 \otimes \cdots \tilde{\mathcal{C}}_m // G$, where $\tilde{\mathcal{C}}_i$ is the space of monodromy data at a_i and \otimes denotes the fusion product between quasi Hamiltonian G -manifolds [3]. In the simple pole case, it recovers the quasi-Hamiltonian description of moduli spaces of flat connections in [3]. The main result of [10] [12] leads to that the irregular Riemann-Hilbert correspondence

$$\nu : (\tilde{O}_1 \times \cdots \tilde{O}_m) // \hookrightarrow (\tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_m) // G$$

associating monodromy/Stokes data to a meromorphic connection on P is a symplectic map. In [9], Boalch studied (a T -reduction version of) the irregular Riemann-Hilbert correspondence in the case of the meromorphic connections have one simple pole and one order two pole. The key feature of this case is that the correspondence gives rise to a Poisson map from the dual of the Lie algebra \mathfrak{g}^* to the dual Poisson Lie group G^* associated to the standard classical r -matrix on \mathfrak{g} .

Now we give the idea of the proof of the main Theorem 1.1. The first step is to find a symplectic geometric interpretation of equation (1), which turns to be a new geometric framework generalizing the Ginzburg-Weinstein linearization. For this purpose, we consider a symplectic slice Σ of T^*G and its Poisson Lie analogue, a symplectic submanifold Σ' of the Lu-Weinstein symplectic double Γ (locally isomorphic to $G \times G^*$) [30]. See Section 2 and the appendix for more details. Then associated to any map $g \in \text{Map}(\mathfrak{g}^*, G)$, we define a local diffeomorphism $F_g : (\Sigma, \omega) \rightarrow (\Sigma', \omega')$. Then a symplectic geometric interpretation of the gauge transformation equation is as follows.

Theorem 1.3. *F_g is a local symplectic isomorphism from (Σ, ω) to (Σ', ω') if and only if $g \in \text{Map}(\mathfrak{g}^*, G)$ satisfies equation (1).*

With the help of the above theorem, we only need to prove the expected symplectic geometry property of the connection matrix C . This is immediate as long as we consider the irregular Riemann-Hilbert correspondence in the setting of the extended moduli space (see Definition 2.6 in [10]) of meromorphic connections with one simple pole and one order two pole. Actually, following the discussion above, the corresponding irregular Riemann-Hilbert map is

$$\nu : (\tilde{O}_1 \times \tilde{O}_2) // G \hookrightarrow (\tilde{\mathcal{C}}_1 \times \tilde{\mathcal{C}}_2) // G.$$

On the other hand, the Hamiltonian and quasi-Hamiltonian quotient $(\tilde{O}_1 \times \tilde{O}_2) // G$ and $(\tilde{\mathcal{C}}_1 \times \tilde{\mathcal{C}}_2) // G$ are isomorphic to Σ and Σ' respectively. We thus obtain a symplectic map $\nu : \Sigma \rightarrow \Sigma'$. Next, following the construction of the irregular Riemann-Hilbert map, we prove that ν can be chosen in such a way that for any $(h, \lambda) \in \Sigma \subset T^*G \cong G \times \mathfrak{g}^*$ (via left multiplication)

$$\nu(h, \lambda) = F_C(h, \lambda),$$

where $C(x)$ is the connection matrix of ∇ in (2). Therefore, combining with Theorem 2.2, we prove that the connection matrix $C \in \text{Map}(\mathfrak{g}^*, G)$ satisfies the gauge transformation equation (1). This clarifies the relation between the gauge transformation of dynamical r -matrices and certain irregular Riemann-Hilbert problem. As a byproduct, we give a new description of Lu-Weinstein symplectic groupoid via Alekseev-Meinrenken r -matrix. We also clarify the meaning of the gauge transformation equation in the framework of generalized classical dynamical r -matrix.

The organisation of this paper is as follows. The next section gives the background material and a geometric description of the equation (1). Section 3 defines the connection matrix $C(x)$ of a meromorphic connection ∇ and states that $C : \mathfrak{g}^* \rightarrow G$ gives rise to a solution of (1), i.e., a

gauge transformation from r_0 to r_{AM} . Section 4 discusses the quantum version, i.e., the vertex-IRF transformation equation and formulates a surprising relation between connection matrices and Drinfeld twists. Section 5 gives the background material on the moduli space of meromorphic connections over surfaces and irregular Riemann-Hilbert correspondence in this setting. At the second part of Section 5, we study in details one special case of this correspondence and show that how it gives rise to the equivariant geometric description of the equation (1). Section 6 describes Lu-Weinstein symplectic groupoid via Alekseev-Meinrenken r -matrix. The appendix studies the Poisson structure on the submanifold Σ' of Lu-Weinstein symplectic double and gives a proof of the main theorem in Section (2.1).

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2 Symplectic geometry and gauge transformations of r -matrices

Throughout this paper, let \mathfrak{g} be a complex reductive Lie algebra and $t \in S^2(\mathfrak{g})^{\mathfrak{g}}$ the element corresponding to an invariant inner product on \mathfrak{g} .

First recall that an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r -matrix if $r + r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and r satisfies the classical Yang-Baxter equation:

$$[r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0.$$

Throughout this paper, we will denote by $r_0 := \frac{1}{2}(r - r^{2,1})$ the skew-symmetric part of a classical r -matrix r .

A dynamical analog of a classical r -matrix is as follows. Let $\eta \subset \mathfrak{g}$ be a Lie subalgebra. Then a classical dynamical r -matrix is an η -equivariant map $r : \eta^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ such that $r + r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and r satisfies the dynamical Yang-Baxter equation (CDYBE):

$$\text{Alt}(dr) + [r^{1,2}, r^{1,3}] + [r^{1,2}, r^{2,3}] + [r^{1,3}, r^{2,3}] = 0, \quad (4)$$

where $\text{Alt}(dr(x)) \in \wedge^3 \mathfrak{g}$ is the skew-symmetrization of $dr(x) \in \eta \otimes \mathfrak{g} \otimes \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ for all $x \in \eta^*$.

In the distinguished special case $\eta = \mathfrak{g}$, the Alekseev-Meinrenken dynamical r -matrix $r_{\text{AM}} : \mathfrak{g}^* \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is defined by

$$r_{\text{AM}}(x) = (\text{id} \otimes \phi(\text{ad}_x^\vee))(t), \quad \forall x \in \mathfrak{g}^*,$$

where $x^\vee = (x \otimes \text{id})(t)$ and $\phi(z) := -\frac{1}{z} + \frac{1}{2} \cotanh \frac{z}{2}$, $z \in \mathbb{C} \setminus 2\pi i\mathbb{Z}^*$. Taking the Taylor expansion of ϕ at 0, we see that $\phi(z) = \frac{z}{12} + o(z^2)$, thus $\phi(\text{ad}_x)$ is well-defined (The maximal domain of definition of $\phi(\text{ad}_x)$ contains all $x \in \mathfrak{g}^*$ for which the eigenvalues of ad_x lie in $\mathbb{C} \setminus 2\pi i\mathbb{Z}^*$). One can check that $r_{\text{AM}} + \frac{t}{2}$ is a classical dynamical r -matrix.

Denote by G the formal group with Lie algebra \mathfrak{g} and by $\text{Map}_0(\mathfrak{g}^*, G)$ the space of formal maps $g : \mathfrak{g}^* \rightarrow G$ such that $g(0) = 1$, i.e., the space of maps of the form e^u , where $u \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{\geq 0}$ ($\hat{S}(\mathfrak{g})$ is the degree completion of the symmetric algebra $S(\mathfrak{g})$). The following theorem states the existence of formal solutions of equation (1).

Theorem 2.1. ([18]) *Let r be a classical r -matrix with $r + r^{2,1} = t$ and $r_0 := \frac{1}{2}(r - r^{2,1})$. Then there exists a formal map $g \in \text{Map}_0(\mathfrak{g}^*, G)$, such that*

$$g_1^{-1}d_2(g_1) - g_2^{-1}d_1(g_2) + (\otimes^2 \text{Ad}_g)^{-1}r_0 + \langle \text{id} \otimes \text{id} \otimes x, [g_1^{-1}d_3(g_1), g_2^{-1}d_3(g_2)] \rangle = r_{\text{AM}}, \quad (5)$$

Here $g_1^{-1}d_2(g)(x) := \sum_i g^{-1} \frac{\partial g}{\partial \xi^i}(x) \otimes e_i$ is viewed as a formal function $\mathfrak{g}^* \rightarrow \mathfrak{g}^{\otimes 2}$, $\{e_i\}$ is a basis of \mathfrak{g} , $\{\xi^i\}$ the corresponding coordinates on \mathfrak{g}^* and $g_i^{-1}d_j(g_i) = (g_1^{-1}d_2(g_1))^{i,j}$.

We will call equation (5) as the gauge transformation equation, and denote its left hand side by $r_0^g \in \text{Map}(\mathfrak{g}^*, \mathfrak{g} \wedge \mathfrak{g})$. In [18], this equation is proven to be the classical limit of vertex-IRF transformation between certain dynamical twists (see section 4) and the authors give two constructions of the formal solutions of equation (5) based on formal calculation and quantization of Lie bialgebras respectively. In the following two sections, we will give a geometric interpretation and construct explicit solutions of equation (5), where instead of the formal setting, we will work on a local theory.

2.1 Geometric construction

The symplectic manifold (Σ, ω) . Following the convention from last section, and let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra and \mathfrak{t}' the complement of the affine root hyperplanes: $\mathfrak{t}' := \{\Lambda \in \mathfrak{t} \mid \alpha(\Lambda) \notin \mathbb{Z}\}$. In the following, \mathfrak{t}' is regarded as a subspace of \mathfrak{g}^* via the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by inner product. Let Σ be a cross-section of $T^*G \cong G \times \mathfrak{g}^*$ (identification via left multiplication), defined by

$$\Sigma := \{(h, \lambda) \in G \times \mathfrak{g}^* \mid \lambda \in \mathfrak{t}'\}.$$

Then one can check that Σ is a symplectic submanifold of T^*G with the canonical symplectic structure (see [26] Theorem 26.7). The induced symplectic structure ω on Σ is given for any tangents $v_1 = (X_1, R_1), v_2 = (X_2, R_2) \in \mathfrak{g} \times \mathfrak{g}^*$, where $R_1, R_2 \in \mathfrak{t}^*$, at $(h, \lambda) \in \Sigma$ by

$$\omega(v_1, v_2) = \langle R_1, X_2 \rangle - \langle R_2, X_1 \rangle + \langle \lambda, [X_1, X_2] \rangle. \quad (6)$$

The Poisson-Lie analogue of (Σ, ω) . Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a classical r -matrix with $r + r^{2,1} = t$. Let G^* be the simply connected dual Poisson Lie group associated to the quasitriangular Lie bialgebra (\mathfrak{g}, r) and D the double Lie group with Lie algebra $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$ which is locally diffeomorphic to $G \times G^*$ (see e.g [29]). A natural symplectic structure on D is given by the following bivector,

$$\pi_D = \frac{1}{2}(r_d \pi_0 + l_d \pi_0),$$

where $\pi_0 \in \mathfrak{d} \wedge \mathfrak{d}$ such that $\pi_0(\xi_1 + X_1, \xi_2 + X_2) = \langle X_1, \xi_2 \rangle - \langle X_2, \xi_1 \rangle$ for $\xi_1 + X_1, \xi_2 + X_2 \in \mathfrak{d}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}$.

Following [30], the Lu-Weinstein double symplectic groupoid, associated to the Lie bialgebra (\mathfrak{g}, r) , is the set

$$\Gamma := \{(h, h^*, u, u^*) \mid h, u \in G, h^*, u^* \in G^*, hh^* = u^*u \in D\}$$

with a unique Poisson structure π_Γ such that the local diffeomorphism $(\Gamma, \pi_\Gamma) \rightarrow (D, \pi_D): (h, h^*, u, u^*) \mapsto hh^*$ is Poisson. We define a submanifold Σ' of Γ , as a Poisson Lie analogue of Σ , by

$$\Sigma' := \{(h, h^*, u, u^*) \in \Gamma \mid h^* \in e^{\mathfrak{t}'} \subset G^*\}$$

(e denotes the exponential map with respect to the Lie algebra \mathfrak{g}^*). In the appendix, we will prove that Σ' is a symplectic submanifold of (Γ, π_Γ) . Now let us take this fact and denote the induced symplectic structure on Σ' by ω' . On the other hand, the map

$$\Sigma' \rightarrow G \times e^{\mathfrak{t}'}; (h, e^\lambda, u, u^*) \mapsto (h, e^\lambda)$$

expresses Σ' as a cover of a dense subset of $G \times e^{\mathfrak{t}'} \subset G \times G^*$. Thus Σ and Σ' are locally diffeomorphic to each other.

Symplectic maps between (Σ, ω) and (Σ', ω') . Associated to any $g \in \text{Map}_0(\mathfrak{g}^*, G)$, we define a map $F_g : \Sigma \rightarrow \Sigma'$ by

$$F_g(h, \lambda) := (g(\text{Ad}_h \lambda)h, e^\lambda, u, u^*), \quad \forall (h, \lambda) \in \Sigma, \quad (7)$$

where $u \in G, u^* \in G^*$ are determined by the identity $g(\text{Ad}_h \lambda)he^\lambda = u^*u$ (understood to hold in the double Lie group D). Note that F_g is well-defined for the elements $(h, \lambda) \in \Sigma$ sufficiently near $(e, 0) \in G \times \mathfrak{t}^*$. This is because for these (h, λ) , $g(\text{Ad}_h \lambda)he^\lambda$ in the double Lie group D is sufficiently near the unit, thus $g(\text{Ad}_h \lambda)he^\lambda = u^*u$ uniquely determines u and u' . So we can think of F_g defined on a local chart and this is enough for our purpose.

Theorem 2.2. *F_g is a local symplectic isomorphism from (Σ, ω) to (Σ', ω') if and only if $g \in \text{Map}(\mathfrak{g}^*, G)$ satisfies the gauge transformation equation (5), $r_0^g = r_{\text{AM}}$.*

Proof. See the appendix.

The case when r is a standard r -matrix. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let B_\pm denote a pair of opposite Borel subgroups with $B_+ \cap B_- = T$. For the choice of positive roots Φ_+ corresponding to Borel subgroup B_+ , we take the standard r -matrix given by

$$r := \frac{1}{2}t + \frac{1}{2} \sum_{\alpha \in \Phi_+} E_\alpha \wedge E_{-\alpha}, \quad (8)$$

where $t \in S^2(\mathfrak{g})^\mathfrak{g}$ is the Casimir element. In this case, the simply connected dual Poisson Lie group associated to (\mathfrak{g}, r) is

$$G^* = \{(b_-, b_+, \Lambda) \in B_- \times B_+ \times \mathfrak{t} \mid \delta(b_-)\delta(b_+) = 1, \delta(b_+) = \exp(\pi i \Lambda)\},$$

where $\delta : \mathfrak{g} \rightarrow \mathfrak{t}$ is the projection corresponding to the root space decomposition. Thus Σ' is a submanifold of the double

$$\Gamma : \{h, (b_-, b_+, \Lambda_x), u, (c_-, c_+, \Lambda_c) \mid hb_\pm = c_\pm u\} \subset (G \times G^*)^2,$$

defined by

$$\Sigma' := \{(h, (e^{-\pi i \Lambda}, e^{\pi i \Lambda}, \Lambda), u, (c_+, c_-, \Lambda_c)) \in \Gamma \mid he^{\pm \pi i \Lambda} = c_\pm u, \Lambda \in \mathfrak{t}'\},$$

where $\mathfrak{t}' \subset \mathfrak{t}$ is the complement of the affine root hyperplanes. To simplify notation, we will write $e^{2\pi i\lambda} \in G^*$ instead of $(e^{\pi i\lambda^\vee}, e^{-\pi i\lambda^\vee}, \lambda^\vee) \in G^*$, where $\lambda \in \mathfrak{g}^*$, $\lambda^\vee = (\lambda \otimes \text{id})(t) \in \mathfrak{g}$ and $e^{2\pi i\lambda}$ (resp. $e^{\pi i\lambda^\vee}$) takes the exponential map of the Lie algebra \mathfrak{g}^* (resp. \mathfrak{g}).

Now given any $g \in \text{Map}(\mathfrak{g}^*, G)$, let us consider a local diffeomorphism $F'_g : \Sigma \rightarrow \Sigma'$ (a rescale of F_g) which will be more directly involved in the following discussion,

$$F'_g(h, \lambda) := (g(2\pi i \text{Ad}_h \lambda)h, e^{2\pi i\lambda}, u, u^*), \quad \forall (h, \lambda) \in \Sigma$$

where $u \in G, u^* \in G^*$ are uniquely determined by the identity $he^\lambda = u^*u$. It is obvious that the map $(\Sigma, \omega) \rightarrow (\Sigma, \frac{1}{2\pi i}\omega)$, $(h, \lambda) \mapsto (h, 2\pi i\lambda)$ is symplectic. Therefore, as a corollary of Theorem 2.2, we have

Corollary 2.3. *The map F'_g is a local symplectic isomorphism from (Σ, ω) to (Σ', ω') (provided the symplectic structure on the right-hand side is divided by $2\pi i$) if and only if g satisfies the gauge transformation equation (5), $r_0^g = r_{\text{AM}}$.*

3 Gauge transformations via Stokes phenomenon

Let G be a complex reductive Lie group, $T \subset G$ a maximal torus, and $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebras of T and G respectively. Let $\Phi \subset \mathfrak{t}^*$ be the corresponding root system of \mathfrak{g} , and $\mathfrak{t}_{\text{reg}}$ the set of regular elements in \mathfrak{t} .

Let P be the holomorphically trivial principal G -bundle on \mathbb{P}^1 . We consider the following meromorphic connection on P of the form

$$\nabla := d - \left(\frac{A_0}{z^2} + \frac{x}{z} \right) dz,$$

where $A_0, x \in \mathfrak{g}$. We assume henceforth that $A_0 \in \mathfrak{t}_{\text{reg}}$ and once fixed, the only variable is $x \in \mathfrak{g} \cong \mathfrak{g}^*$ (via the inner product on \mathfrak{g}). Note that the connection ∇ has an order 2 pole at origin and (if $x \neq 0$) a first order pole at ∞ .

Definition 3.1. *The Stokes rays of the connection ∇ are the rays $\mathbb{R}_{>0} \cdot \alpha(A_0) \subset \mathbb{C}^*$, $\alpha \in \Phi$. The Stokes sectors are the open regions of \mathbb{C}^* bounded by them.*

Let us choose an initial sector Sect_0 at 0 bounded by two adjacent Stokes rays and a branch of $\log(z)$ on Sect_0 . Then we label the Stokes rays d_1, d_2, \dots, d_{2l} going in a positive sense and starting on the positive edge of Sect_0 . Set $\text{Sect}_i = \text{Sect}(d_i, d_{i+1})$ for the open sector bounded by the rays d_i to d_{i+1} . (Indices are taken modulo $2l$, so $\text{Sect}_0 = \text{Sect}(d_{2l}, d_1)$).

To each sector Sect_i , there is a canonical fundamental solution F_i of ∇ with prescribed asymptotics in the i -th supersector $\widehat{\text{Sect}}_i := \text{Sect}(d_i - \frac{\pi}{2}, d_i + \frac{\pi}{2})$. In particular, the following result is proved in [8] for $G = GL_n(\mathbb{C})$, in [11] for G reductive, and in [14] for an arbitrary affine algebraic group. Denote by $\delta(x)$ the projection of x onto \mathfrak{t} corresponding to the root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

Theorem 3.2. *On each sector Sect_i , there is a unique holomorphic function $H_i : \text{Sect}_i \rightarrow G$ such that the function*

$$F_i = H_i \cdot e^{-\frac{A_0}{z}} \cdot z^{\delta(x)}$$

satisfies $\nabla F_i = 0$, and H_i can be analytically continued to $\widehat{\text{Sect}}_i$ and then H_i is asymptotic to 1 within $\widehat{\text{Sect}}_i$.

3.1 Connection matrix and dynamical r-matrices

The meromorphic connection $\nabla = d - (\frac{A_0}{z^2} + \frac{x}{z})dz$ is said to be *non-resonant* at $z = \infty$ if the eigenvalues of $\text{ad}(x)$ are not positive integers. The following fact is well-known (see e.g [35] for $G = \text{GL}_n(\mathbb{C})$).

Lemma 3.3. *If ∇ is non-resonant, there is a unique holomorphic function $H_\infty : \mathbb{P}^1 \setminus \{0\} \rightarrow G$ such that $H_\infty(\infty) = 1$, and the function $F_\infty = H_\infty \cdot z^B$ is a solution of $\nabla F = 0$.*

Now let us consider the solutions of $\nabla F = 0$:

$$\begin{aligned} F_0 & \text{ on } \text{Sect}_0, \\ F_\infty & = H_\infty \cdot z^x \text{ on a neighbourhood of } \infty \text{ slit along } d_1, \end{aligned}$$

We define the **connection matrix** $C(x) \in G$ (with respect to the chosen Sect_0) by

$$F_\infty = F_0 \cdot C(x).$$

Here F_∞ is extended along a path in Sect_0 then the identity is understood to hold in the domain of definition of F_0 .

Thus we obtain a map $C_{2\pi i} : \mathfrak{g}_{\text{nr}}^* \rightarrow G$ (depends on the choice of A_0) which maps any $x \in \mathfrak{g}_{\text{nr}}^*$ to the connection matrix $C(\frac{1}{2\pi i}x) \in G$ of $\nabla = d - (\frac{A_0}{z^2} + \frac{1}{2\pi i}\frac{x}{z})dz$. Here $\mathfrak{g}_{\text{nr}}^* \subset \mathfrak{g}^*$ is the dense open set corresponding to the set of elements x such that the eigenvalues of $\frac{1}{2\pi i}\text{ad}(x)$ do not contain positive integers (provided we identify $\mathfrak{g} \cong \mathfrak{g}^*$).

Note that the Stokes sector Sect_0 determines a partition of the root system Φ of \mathfrak{g} as follows. Let Π_+ and Π_- be the sets of Stokes rays which one crosses when going from Sect_0 to the opposite sector Sect_l in the counterclockwise and clockwise directions respectively. Then $\Phi = \Phi_+ \sqcup \Phi_-$, where

$$\Phi_\pm = \{\alpha \in \Phi \mid \alpha(A_0) \in \ell, \ell \in \Pi_\pm\} = -\Phi_\mp.$$

Now let us consider the equation (5), in which r_0 is the skew-symmetric part of the standard r -matrix associated to the positive root system Φ_+ . Our main theorem states that

Theorem 3.4. *The map $C_{2\pi i} \in \text{Map}(\mathfrak{g}_{\text{nr}}^*, G)$ is a solution of the gauge transformation equation (5).*

A proof will be given in Section 5. The idea is as follows. Following Theorem 2.2, to prove $r_0^C = r_{\text{AM}}$, we only need to verify its symplectic geometric counterpart, i.e., $F'_C : (\Sigma, \omega) \rightarrow (\Sigma', \omega')$ is a symplectic map. The map F'_C will be realized as certain irregular Riemann-Hilbert map in section 5, and then F'_C is symplectic following from the symplectic nature of the irregular Riemann-Hilbert correspondence.

3.2 Stokes matrices and linearization of G^*

Given the initial Stokes sector Sect_0 and the determination of $\log(z)$ with a cut along the Stokes ray d_1 , the Stokes matrices are essentially the transition matrices between the canonical solutions F_0 on Sect_0 and F_l on the opposite sector Sect_l , when they are continued along the two possible paths in the punctured disk joining these sectors. Thus the *Stokes matrices* of $\nabla = d - (\frac{A_0}{z^2} + \frac{x}{z})dz$ with respect to Sect_0 are the elements $S_\pm(x)$ of G determined by

$$F_0 = F_l \cdot S_+(x)e^{2\pi i\delta(x)}, \quad F_l = F_0 \cdot S_-(x)$$

where $\delta(x)$ takes the projection of \mathfrak{g} onto \mathfrak{t} , and the first (resp. second) identity is understood to hold in Sect_l (resp. Sect_0) after F_0 (resp. F_l) has been analytically continued counterclockwise. The connection matrix $C(x)$ is related to the Stokes matrices $S_{\pm}(x)$ by the following monodromy relation (from the fact that a simple positive loop around 0 is a simple negative loop around ∞).

Lemma 3.5. *The following holds*

$$C(x)^{-1}e^{2\pi i\delta(x)}C(x) = S_-(x)S_+(x)e^{2\pi i\delta(x)}$$

Recall that the Stokes sector Sect_0 determines a partition of the root system $\Phi = \Phi_+ \sqcup \Phi_-$. Let $U_{\pm} \subset G$ be the unipotent subgroups with Lie algebra $\mathfrak{u}_{\pm} = \bigoplus_{\alpha \in \Phi_{\pm}} \mathfrak{g}_{\alpha}$, and B_{\pm} the corresponding opposite Borel subgroups. It follows from [11] that the Stokes matrices $S_+(x), S_-(x)$ lie in U_+, U_- respectively. Varying $x \in \mathfrak{g}^*$, we therefore obtain the Stokes map

$$\mu : \mathfrak{g}^* \rightarrow G^*; x \mapsto (e^{-\pi i\delta(x)}S_-(x)^{-1}, e^{-\pi i\delta(x)}S_+(x)e^{2\pi i\delta(x)}, e^{\pi i\delta(x)}).$$

Here G^* is the dual Poisson Lie group defined in Section 2,

$$G^* = \{(b_-, b_+, \Lambda) \in B_- \times B_+ \times \mathfrak{t} \mid \delta(b_-)\delta(b_+) = 1, \delta(b_+) = \exp(\pi i\Lambda)\}.$$

The relation between the Stokes map μ and the theory of Poisson Lie groups can be shown as follows.

It follows from Theorem 3.4 and Corollary 2.3 that the map

$$F'_C : \Sigma \rightarrow \Sigma'; (h, \lambda) \mapsto (C(\text{Ad}_h \lambda)h, e^{2\pi i\lambda}, u, u^*)$$

is a local symplectic isomorphism. This map is equivariant with respect to the symplectic T -actions on Σ and Σ' , which are respectively given by

$$a \cdot (h, \lambda) = (ha, \lambda), \quad a \cdot (h, e^{\lambda}, u, u^*) = (ha, e^{\lambda}, u, u^*), \quad \forall a \in T.$$

Define two maps $P : \Sigma \rightarrow \mathfrak{g}^*, P' : \Sigma' \rightarrow G^*$ whose fibres are the T orbits

$$P(h, \lambda) = \text{Ad}_h^* \lambda, \quad \forall (h, \lambda) \in \Sigma, \quad P'(h, e^{\lambda}, u, u^*) = d_h e^{\lambda}, \quad \forall (h, e^{\lambda}, u, u^*) \in \Sigma'.$$

Here d denotes the left dressing transformation of G on G^* . By using the monodromy relation (3.5), we check that the Stokes map $\mu : \mathfrak{g}^* \rightarrow G^*$ is the unique map such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \xrightarrow{F'_C} & \Sigma' \\ P_1 \downarrow & & \downarrow P_2 \\ \mathfrak{g}^* & \xrightarrow{S_C} & G^* \end{array}$$

One checks that P and P' are Poisson maps (see e.g the appendix), where \mathfrak{g}^* is equipped with its standard Kirillov–Kostant–Souriau Poisson structure and G^* the dual Poisson Lie group structure. Therefore, the T -reduction of the symplectic isomorphism F'_C gives rise to the following remarkable result due to Boalch.

Theorem 3.6. [9] *The Stokes map $\mu : \mathfrak{g}^* \rightarrow G^*$ is a local analytic Poisson isomorphism.*

4 Vertex-IRF transformations, Drinfeld twists and Stokes phenomenon

Following [18], the gauge equation (5) is the classical limit of a vertex-IRF transformation equation [21]. In particular, a Drinfeld twist killing an admissible associator gives rise to such a vertex-IRF transformation. Therefore the classical limit of such a twist is a solution of equation (5). Thus we have two sources of solutions (as is shown in last section, another one is from Stokes phenomenon). In this section, we will prove that there exists a Drinfeld twist whose classical limit is the connection matrix C introduced in last section.

Let $(U(\mathfrak{g}), m, \Delta, \varepsilon)$ denote the universal enveloping algebra of \mathfrak{g} with the product m , the co-product Δ and the counit ε . Let $U(\mathfrak{g})[[\hbar]]$ be the corresponding topologically free $\mathbb{C}[[\hbar]]$ -algebra.

Definition 4.1. Let $\Phi = 1 + \frac{[t^{12}, t^{23}]}{24} \hbar^2 + O(\hbar^3) \in U(\mathfrak{g})^{\hat{\otimes}^3}[[\hbar]]$ be such that Φ is \mathfrak{g} -invariant and satisfies the pentagon equation and the counit axiom. Then a function $J_d : \mathfrak{g}^* \rightarrow U(\mathfrak{g})^{\hat{\otimes}^2}[[\hbar]]$ is called a dynamical twist associated to Φ if $J_d(x) = 1 + O(\hbar)$ is \mathfrak{g} -invariant and

$$J_d^{12,3} J_d^{1,2}(x + \hbar h^{(3)}) = \Phi^{-1} J_d^{1,23}(x) J_d^{2,3}(x), \quad (9)$$

where for $J_d^{1,2}(x + \hbar h^{(3)})$ we use the dynamical convention, i.e.,

$$J_d^{1,2}(x + \hbar^3) = \sum_{N \geq 0} \frac{\hbar^N}{N!} \sum_{i_1, \dots, i_N}^n (\partial_{\xi^{i_1}} \cdots \partial_{\xi^{i_N}} J_d)(x) \otimes (e_{i_1} \cdots e_{i_N})$$

where $n = \dim(\mathfrak{g})$, and $\{e_i\}_{i=1, \dots, n}$, $\{\xi^i\}_{i=1, \dots, n}$ are dual bases of \mathfrak{g} and \mathfrak{g}^* .

Assume $(\Phi, J_d(x))$ satisfies the conditions in Definition 4.1. Let $j(x) := (\frac{J_d(x) - 1}{\hbar}) \bmod \hbar$, and $r(x) := j(x) - j(x)^{2,1}$. Then following [17], $r(x) + \frac{t}{2}$ is a solution of the CDYBE (4), i.e., a classical dynamical r -matrix. In this case, $J_d(x)$ is called a dynamical twist quantization of $r(x)$. In particular, a constant twist $J_c \in U(\mathfrak{g})^{\hat{\otimes}^2}[[\hbar]]$ is such that

$$J_c^{12,3} J_c^{1,2} = \Phi^{-1} J_c^{1,23} J_c^{2,3}. \quad (10)$$

Similarly, we say $J_c = 1 + \hbar \frac{r}{2} + o(\hbar^2)$ is a twist quantization of $r_0 := \frac{1}{2}(r - r^{2,1})$.

Set $U' := U(\hbar \mathfrak{g})[[\hbar]]$, the subalgebra generated by $\hbar x$, $\forall x \in \mathfrak{g}$. Note that $U'/\hbar U' = \hat{S}(\mathfrak{g})$. An associator $\Phi \in U(\mathfrak{g})^{\hat{\otimes}^3}[[\hbar]]$ is called admissible (see [19]) if

$$\Phi \in 1 + \frac{\hbar^2}{24} [t^{1,2}, t^{2,3}] + O(\hbar^3), \quad \hbar \log(\Phi) \in (U'(\mathfrak{g}))^{\hat{\otimes}^3}.$$

Given an admissible associator $\Phi \in U(\mathfrak{g})^{\hat{\otimes}^3}[[\hbar]]$, we identify the third component $U(\mathfrak{g})$ of this tensor cube with $\mathbb{C}[\mathfrak{g}^*]$ via the symmetrization (PBW) isomorphism $S : (\mathfrak{g}) \rightarrow U(\mathfrak{g})$ and use this identification view Φ^{-1} as a function from \mathfrak{g}^* to $U(\mathfrak{g})^{\hat{\otimes}^2}[[\hbar]]$, denoted by $\Phi^{-1}(x)$. Then we have $\Phi^{-1}(\hbar^{-1}x)$ is a well-defined element in $U(\mathfrak{g})^{\hat{\otimes}^2} \hat{\otimes} \mathbb{C}[\mathfrak{g}^*]$. Following [19], any universal Lie associator gives rise to an admissible associator.

Theorem 4.2. [17] Assume that Φ is the image in $U(\mathfrak{g})^{\hat{\otimes}^3}[[\hbar]]$ of a universal Lie associator. Let $J_d(x) := \Phi^{-1}(\hbar^{-1}x)$, where Φ^{-1} is regarded as an element of $(U(\mathfrak{g})^{\hat{\otimes}^2} \otimes \mathbb{C}[\mathfrak{g}^*])[[\hbar]]$. Then

(1) $J_d(x)$ is a formal dynamical twist. More precisely, $J_d(x) = 1 + \hbar j(x) + O(\hbar^2) \in (U(\mathfrak{g})^{\hat{\otimes} 2} \hat{\otimes} \hat{S}(\mathfrak{g}))[[\hbar]]$, is a series in nonnegative powers of \hbar and satisfies the dynamical twist equation.

(2) $J_d(x)$ is a twist quantization of the Alekseev-Meinrenken dynamical r -matrix, that is $r_{\text{AM}} = j(x) - j(x)^{2,1}$.

(3) If Φ_{KZ} is the Knizhnik-Zamolodchikov associator, then $J_d(x)$ is holomorphic on an open set and extends meromorphically to the whole \mathfrak{g}^* .

Definition 4.3. [21] Let $J_d(x) : \mathfrak{g}^* \rightarrow U(\mathfrak{g})^{\hat{\otimes} 2}[[\hbar]]$ be a function with invertible values and $\rho : \mathfrak{g}^* \rightarrow U(\mathfrak{g})[[\hbar]]$ a function with invertible values such that $\varepsilon(\rho(\lambda)) = 1$ (ε is the counit). Set

$$J_d^\rho(x) = \Delta(\rho(x))J_d(x)\rho^1(x - \hbar h^{(2)})^{-1}\rho^2(x)^{-1},$$

and call ρ a vertex-IRF transformation from $J_d(x)$ to $J_d^\rho(x)$, where for $\rho^{-1}(x - \hbar h^{(2)})$ we use the dynamical convention.

Now let us take an admissible associator Φ . Let J_c (resp. $J_d(x)$) be a (resp. dynamical) twist quantization of r_0 (resp. r_{AM}). Let $\rho(x) \in (U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}))[[\hbar]]$ be a formal vertex-IRF transformation which maps the \mathfrak{g} -invariant but dynamical twist $J_d(x)$ to the constant but non-invariant twist J_c . This is to say

$$J_c = \Delta(\rho(x))J_d(x)\rho^1(x - \hbar h^{(2)})^{-1}\rho^2(x)^{-1}. \quad (11)$$

Then by comparing the coefficients of equation (11) up to the first order of \hbar , we have

Proposition 4.4. [18] The reduction modulo \hbar of $\rho(x)$, denoted by $g(x) = \rho(x)|_{\hbar=0}$, belongs to $\exp(\mathfrak{g} \otimes \hat{S}(\mathfrak{g}))_{>0}$ (thus a formal map from \mathfrak{g}^* to $\exp(\mathfrak{g})$) and satisfies the equation $r_0^{g(x)} = r_{\text{AM}}$.

Let $J_d(x) = \Phi(\hbar^{-1}x)$ be the dynamical twist in Theorem 4.2, then IRF-transformations satisfying (11) are constructed in [18] as follows. For the admissible associator Φ , there exists a twist killing Φ (see [15][20]), and according to [19], this twist can be made admissible by a suitable gauge transformation. The resulting twist $J \in U(\mathfrak{g})^{\hat{\otimes} 2}[[\hbar]]$ satisfies $J = 1 - \hbar \frac{\tau}{2} + o(\hbar)$, $\hbar \log(J) \in U^{\prime \hat{\otimes} 2}$, $(\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1$, and

$$\Phi = (J^{2,3} J^{1,23})^{-1} J^{1,2} J^{12,3}. \quad (12)$$

Let us now identify the second component $U(\mathfrak{g})$ of J with $\mathbb{C}(\mathfrak{g}^*)$ via PBW isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$, and regard J as a formal function from \mathfrak{g}^* to $U(\mathfrak{g})[[\hbar]]$, denoted by $J(x)$. Let $\rho(x) := J(\hbar^{-1}x) \in \text{Map}(\mathfrak{g}^*, U(\mathfrak{g})[[\hbar]])$ denote the corresponding renormalization by sending $x \in \mathfrak{g}^*$ to $\hbar^{-1}x$. Then if we identify the third component $U(\mathfrak{g})$ of the tensor cube with $\mathbb{C}(\mathfrak{g}^*)$ in equation (11) and renormalize the resulting formal maps from \mathfrak{g}^* to $U(\mathfrak{g})^{\hat{\otimes} 2}$ by sending $x \in \mathfrak{g}^*$ to $\hbar x$, the equation (12) becomes

$$J^{-1} = \Delta(\rho(x))J_d(x)\rho^1(x - \hbar h^{(2)})^{-1}\rho^2(x)^{-1}$$

(Here $J_d(x) := \Phi^{-1}(\hbar^{-1}x)$ is the dynamical twist as in Theorem 4.2). One checks that $J_c := J^{-1}$ satisfies (10) (thus a constant twist). Therefore, the admissible Drinfeld twist J gives rise to a vertex-IRF transformation between the dynamical twist $J_d(x)$ and the constant twist $J_c = J^{-1}$. For an admissible Drinfeld twist J , regarded as a formal function $J(x) : \mathfrak{g}^* \rightarrow U(\mathfrak{g})[[\hbar]]$ (via PBW), define the renormalized classical limit of J by $g(x) := J(\hbar^{-1}x)|_{\hbar=0} \in \text{Map}(\mathfrak{g}^*, U(\mathfrak{g})[[\hbar]])$ (by Proposition 4.4 $g(x)$ is actually in $\text{Map}_0(\mathfrak{g}^*, G)$).

Following Proposition 4.4 and the above discussion, the renormalized classical limit $g(x) \in \text{Map}(\mathfrak{g}^*, G)$ of an admissible Drinfeld twist J satisfies equation (5). For the case \mathfrak{g} is semisimple, we will prove that the inverse is also true, i.e., given any solution $g(x)$ of (5), there exists an admissible Drinfeld twist J whose classical limit is $g(x)$. In order to prove this, we need to consider the gauge action on the Drinfeld twist and, as a classical limit, on the space of solutions of (5).

Gauge actions on Drinfeld twists. Recall that $U' = U(\hbar\mathfrak{g})[[\hbar]]$, the subalgebra generated by $\hbar x$, $\forall x \in \mathfrak{g}$. Let $U'_0 := \text{Ker}(\varepsilon) \cap U'$. Then $V := \{u_{\hbar} \in \hbar^{-1}U'_0 \subset U(\mathfrak{g})[[\hbar]] \mid u_{\hbar} = O(\hbar)\}$ is a Lie subalgebra for the commutator. One checks that $e^{u_{\hbar}} * J := (e^{u_{\hbar}})^1 (e^{u_{\hbar}})^2 J (\Delta(e^{u_{\hbar}}))^{-1}$ is a solution of (12) if J is. Thus V acts on the set of admissible Drinfeld twists by $\delta_{u_{\hbar}}(J) = u_{\hbar}^1 J + u_{\hbar}^2 J - J u_{\hbar}^{12}$, $u_{\hbar} \in V$. Note that $V/\hbar V = (\hat{S}(\mathfrak{g})_{>1}, \{-, -\})$. Let $u \in \hat{S}(\mathfrak{g})_{>1}$ be the corresponding classical limit of $u_{\hbar} \in V$, and $g \in \exp(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g})_{>0}$ the classical limit of J . Then the reduction modulo \hbar of this infinitesimal gauge action is given by

$$\delta_u(g) = \{1 \otimes u, g\} - g \cdot du, \quad (13)$$

where $du := e_i \otimes \frac{\partial u}{\partial \xi^i} \in (\mathfrak{g}) \otimes \hat{S}(\mathfrak{g})$ for an orthogonal basis $\{e_i\}$ of \mathfrak{g} and the corresponding coordinates $\{\xi^i\}$ on \mathfrak{g}^* . This infinitesimal gauge action has a geometric description as follows.

Gauge actions on the space of solutions of (5). Let $\text{Map}_0(\mathfrak{g}^*, G)$ be the space of formal maps $g : \mathfrak{g}^* \rightarrow G$ such that $g(0) = 1$. Let us introduce a group structure on $\text{Map}_0(\mathfrak{g}^*, G)$, defined by $(g_1 * g_2)(x) := g_2(\text{Ad}_{g_1(x)}^* x) g_1(x)$. Then there is a natural group homomorphism

$$\text{Map}_0(\mathfrak{g}^*, G) \rightarrow \text{Diff}(\mathfrak{g}^*),$$

which maps $g \in \text{Map}_0(\mathfrak{g}^*, G)$ to the obvious diffeomorphism $g \cdot x = \text{Ad}_{g(x)} x$, $\forall x \in \mathfrak{g}^*$. Let us take the subgroup $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ whose elements, under the above group homomorphism, correspond to Poisson isomorphisms on \mathfrak{g}^* (equipped with its canonical linear Poisson structure). Explicitly, the elements g of $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ are such that (use the same convention in (5))

$$g_1^{-1} d_2(g_1) - g_2^{-1} d_1(g_2) + \langle \text{id} \otimes \text{id} \otimes x, [g_1^{-1} d_3(g_1), g_2^{-1} d_3(g_2)] \rangle = 0.$$

Then it is direct to check that $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ is a pronipotent Lie group with Lie algebra $\{\alpha \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{\geq 1} \mid \text{Alt}(d\alpha) = 0\}$. This Lie algebra is isomorphic to $(\hat{S}(\mathfrak{g})_{>1}, \{-, -\})$ under the map $d : u \rightarrow du \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{\geq 1}$, for all $f \in \hat{S}(\mathfrak{g})_{>1}$.

The right action of $\text{Map}_0(\mathfrak{g}^*, G)$ on itself restricts to an action of $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ on the space of solutions of equation (5). It induces a natural gauge action of $\alpha \in \text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ on the space of solutions g , $(\alpha * g)(x) = g(\text{Ad}_{\alpha(x)}^* x) \alpha(x)$. The infinitesimal of this action is that each $u \in (\hat{S}(\mathfrak{g})_{>1}, \{-, -\})$ (Lie algebra of $\text{Map}_0(\mathfrak{g}^*, G)$) acts by vector fields on the space of solutions by

$$g^{-1} \delta_u(g) = \langle \text{id} \otimes \text{id} \otimes x, [d_3(u_2), g_{12}^{-1} d_3 g_{12}] \rangle - du \in \mathfrak{g} \otimes \hat{S}(\mathfrak{g})_{\geq 0}. \quad (14)$$

It coincides with the infinitesimal gauge action (13). Therefore we have a commutative diagram of gauge actions and taking classical limit

$$\begin{array}{ccc} J & \xrightarrow{e^{u_{\hbar}}} & e^{u_{\hbar}} * J \\ c.l \downarrow & & c.l \downarrow \\ g & \xrightarrow{e^u} & e^u * g \end{array}$$

Here we assume the classical limit (c.l) of J (resp. $u_{\hbar} \in V$) is g (resp. $u \in \hat{S}(\mathfrak{g})_{>1}$), and e^u is seen as an element in $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ under the Lie algebra isomorphism $\hat{S}(\mathfrak{g})_{>1} \cong \text{Lie}(\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G))$. This fact enables us to prove the following Proposition.

Proposition 4.5. *Given any $g \in \text{Map}_0(\mathfrak{g}^*, G)$ satisfying $r_0^g = r_{\text{AM}}$, there exists an admissible Drinfeld twist J whose renormalized classical limit is $g(x)$, and the identity $r_0^g = r_{\text{AM}}$ is the classical limit of the identity $\Phi = (J^{2,3} J^{1,23})^{-1} J^{1,2} J^{12,3}$.*

Proof. Let J' be an admissible Drinfeld twist with $g'(x)$ as its renormalization classical limit (thus a solution of (5)). Following [18], $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$ acts simply and transitively on the space of solutions of (5). Therefore, the two solutions g and g' are related by a map $\alpha \in \text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$, i.e., $g(x) = g' * \alpha$. Let us assume $u \in \hat{S}(\mathfrak{g})_{>1}$ (Lie algebra of $\text{Map}_0^{\text{ham}}(\mathfrak{g}^*, G)$) is such that $e^u = \alpha$, and then take an element $u_{\hbar} \in V \subset U(\mathfrak{g})[[\hbar]]$ whose reduction modulo \hbar is u . The gauge action of $e^{u_{\hbar}}$ on J' provides a new admissible twist $J := e^{u_{\hbar}} * J'$. Furthermore the above commutative diagram verifies $g(x) = J(\hbar^{-1}x)|_{\hbar=0}$ (regard J as a formal function from \mathfrak{g}^* to $U(\mathfrak{g})[[\hbar]]$), i.e., the renormalized classical limit of J is $g(x)$. ■

In particular, given any connection matrix C , $C_{2\pi i} \in \text{Map}_0(\mathfrak{g}^*, G)$ is a solution of (5) (see Section 3.1), therefore can be quantized. From the above discussion, it means that if we regard $C_{2\pi i}$ as an element in $U(\mathfrak{g})^{\otimes 2}$ by taking the Taylor expansion at 0 and identifying $\hat{S}(\mathfrak{g})$ with $U(\mathfrak{g})$, then there exists an admissible Drinfeld twist $J \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ satisfying (12) whose renormalized classical limit is $C_{2\pi i}$.

Theorem 4.6. *Assume Φ is the image in $U(\mathfrak{g})^{\otimes 3}$ of a universal Lie associator. Then for any connection matrix $C \in \text{Map}_0(\mathfrak{g}^*, G)$, there exists an admissible Drinfeld twist J killing the associator Φ whose renormalized classical limit is $C_{2\pi i}$.*

In particular, let Φ be the Knizhnik-Zamolodchikov (KZ) associator Φ_{KZ} , which is constructed as the monodromy from 1 to ∞ of the KZ equation on \mathbb{P}^1 with three simple poles at 0, 1, ∞ . Naively the confluence of two simple poles at 0 and 1 in the KZ equation turns the monodromy representing KZ associator to the monodromy C_{\hbar} representing the connection matrix of certain differential equation with one degree two pole. Recall that the connection matrix C is the monodromy from 0 to ∞ of the equation $\nabla F = 0$. Then the above theorem indicates that the monodromy C_{\hbar} is related to certain Drinfeld twist killing Φ_{KZ} . In other words, the identity $\Phi_{KZ} = (J^{2,3} J^{1,23})^{-1} J^{1,2} J^{12,3}$ may be related to the confluence of simple poles in the KZ equation. The precise relation between Stokes phenomenon and the theory of quantum groups is given in [33] and [34].

5 Irregular Riemann-Hilbert correspondence

In this section, we will recall symplectic moduli spaces of meromorphic connections on a trivial holomorphic principal G -bundle, the corresponding symplectic spaces of monodromy data and the irregular Riemann-Hilbert correspondence between them. We mainly follow the papers [10][11][12] of Boalch, in which these symplectic spaces are found and described both explicitly and from an infinite dimensional viewpoint (generalising the Atiyah-Bott approach). After that, we will consider the case of the meromorphic connections with one simple pole and one order two pole, and prove that the irregular Riemann-Hilbert correspondence in this case gives rise to a gauge transformation between r_0 and r_{AM} .

5.1 Moduli spaces of meromorphic connections and the spaces of monodromy data

Let $D = \sum_{i=1}^m k_i(a_i) > 0$ be an effective divisor on \mathbb{P}^1 and P a holomorphically trivial principal G -bundle. Let us consider the meromorphic connections on P with poles on D . They can be described explicitly as follows. Let z be a local coordinate on \mathbb{P}^1 vanishing at a_i . Then in terms of a local trivialisation of P , any meromorphic connection ∇ on P takes the form of $\nabla = d - A$, where

$$A = \frac{A_{k_i}}{z^{k_i}} dz + \cdots \frac{A_1}{z} dz + A_0 dz + \cdots,$$

and $A_j \in \mathfrak{g}$, $j \leq k_i$.

Definition 5.1. *A compatible framing at a_i of a trivial principal G -bundle P with a generic connection ∇ is an isomorphism ${}^i g_0 : P_{a_i} \rightarrow G$ between the fibre P_{a_i} and G such that the leading coefficient of ∇ is inside $\mathfrak{t}_{\text{reg}}$ in any local trivialisation of P extending g_0 .*

Let us choose, at each point a_i , an irregular type

$${}^i A^0 := {}^i A_{k_i} \frac{dz}{z^{k_i}} + \cdots + {}^i A_2 \frac{dz}{z^2},$$

where ${}^i A_{k_i} \in \mathfrak{t}_{\text{reg}}$ and ${}^i A_j \in \mathfrak{t}$ for $j < k_i$. Let $\nabla = d - A$ in some local trivialisation (thus a compatible framing is an element in G) and z_i a local coordinate vanishing at a_i . Then we say (∇, P) with compatible framing ${}^i g_0$ at a_i has irregular type ${}^i A^0$ if there is some formal bundle automorphism $g \in G[[z_i]]$ with $g(a_i) = {}^i g_0$, such that

$$g[A] := gAg^{-1} + dg \cdot g^{-1} = {}^i A^0 + \frac{{}^i \Lambda}{z_i} dz_i$$

for some ${}^i \Lambda \in \mathfrak{t}$. We denote by \mathbf{a} the choice of the effective divisor $D = \sum_{i=1}^m k_i(a_i)$ and all the irregular types ${}^i A^0$.

Definition 5.2. ([10]) *The extended moduli space $\widetilde{\mathcal{M}}^*(\mathbf{a})$ is the set of isomorphism classes of triples (P, ∇, \mathbf{g}) , consisting of a generic connection ∇ with poles on D on a trivial holomorphic principal G -bundle P on \mathbb{P}^1 with compatible framing $\mathbf{g} = ({}^1 g_0, \dots, {}^m g_0)$, such that (P, ∇, \mathbf{g}) has irregular type ${}^i A^0$ at each a_i .*

Next let us recall (from [10] Section 2) the building blocks \widetilde{O} of the moduli space $\widetilde{\mathcal{M}}^*(\mathbf{a})$. Fix an integer $k \neq 2$. Let $G_k := G(\mathbb{C}[z]/z^k)$ be the group of $(k-1)$ -jets of bundle automorphisms, and let $\mathfrak{g}_k = \text{Lie}(G_k)$ be its Lie algebra, which contains elements of the form $X = X_0 + X_1 z + \cdots + X_{k-1} z^{k-1}$ with $X_i \in \mathfrak{g}$. Let B_k be the subgroup of G_k of elements having constant term 1. The group G_k is the semi-direct product $G \ltimes B_k$ (where G acts on B_k by conjugation). Correspondingly the Lie algebra of G_k decomposes as a vector space direct sum and dualising we have: $\mathfrak{g}_k^* = \mathfrak{b}_k^* \oplus \mathfrak{g}^*$. Elements of \mathfrak{g}_k^* will be written as

$$A = A_k \frac{dz}{z^k} + \cdots + A_1 \frac{dz}{z}$$

via the pairing with \mathfrak{g}_k given by $\langle A, X \rangle := \text{Res}_0(A, X) = \sum_j \langle A_j, X_{j-1} \rangle$. In this way \mathfrak{b}_k^* is identified with the set of A having zero residue and \mathfrak{g}^* with those having only a residue term (zero irregular part). Let $\pi_{\text{res}} : \mathfrak{g}_k^* \rightarrow \mathfrak{g}^*$ and $\pi_{\text{irr}} : \mathfrak{g}_k^* \rightarrow \mathfrak{b}_k^*$ denote the corresponding projections.

Now choose an element $A^0 = A_k^0 \frac{dz}{z^k} + \cdots + A_2^0 \frac{dz}{z^2}$ of \mathfrak{b}_k^* with $A_i^0 \in \mathfrak{t}$ and $A_0^0 \in \mathfrak{t}_{\text{reg}}$. Let $O_{A^0} \subset \mathfrak{b}_k^*$ denote the B_k coadjoint orbit containing A^0 .

Definition 5.3. ([10]) The extended orbit $\tilde{O} \subset G \times \mathfrak{g}_k^*$ associated to O_{A^0} is

$$\tilde{O} := \{(g_0, A) \in G \times \mathfrak{g}_k^* \mid \pi_{irr}(g_0 A g_0^{-1}) \in O_{A^0}\}$$

where $\pi_{irr} : \mathfrak{g}_k^* \rightarrow \mathfrak{b}_k^*$ is the natural projection removing the residue.

\tilde{O} is naturally a Hamiltonian G -manifold. Any tangents v_1, v_2 to $\tilde{O} \in G \times \mathfrak{g}_k^*$ at (g_0, A) are of the form

$$v_i = (X_i(0), [A, X_i] + g_0^{-1} R_i g_0) \in \mathfrak{g} \oplus \mathfrak{g}_k^*$$

for some $X_1, X_2 \in \mathfrak{g}_k$ and $R_1, R_2 \in \mathfrak{t}^*$ (where $\mathfrak{g} \cong T_{g_0}G$ via left multiplication), and the symplectic structure on \tilde{O} is given by

$$\omega_{\tilde{O}}(v_1, v_2) = \langle R_1, \text{Ad}_{g_0} X_2 \rangle - \langle R_2, \text{Ad}_{g_0} X_1 \rangle + \langle A, [X_1, X_2] \rangle. \quad (15)$$

Proposition 5.4. ([10]) The G action $h \cdot (g_0, A) := (g_0 h^{-1}, h A h^{-1})$ on $(\tilde{O}, \omega_{\tilde{O}})$ is Hamiltonian with moment map $\mu_G : \tilde{O} \rightarrow \mathfrak{g}^*$, $\mu(g_0, A) = \pi_{res}(A)$.

In the simple pole case $k = 1$ we define

$$\tilde{O} := \{(h, x) \in G \times \mathfrak{g}^* \mid \text{Ad}_h x \in \mathfrak{t}'\} \subset G \times \mathfrak{g}^*.$$

One checks that the map $\tilde{O} \rightarrow \Sigma$, $(h, x) \mapsto (h, \text{Ad}_h x)$ is an isomorphism of \tilde{O} to the symplectic slice Σ defined in section 3.

The spaces \tilde{O} enable one to construct global symplectic moduli spaces of meromorphic connections on trivial G -bundles over \mathbb{P}^1 as symplectic quotients of the form $\tilde{O}_1 \times \cdots \times \tilde{O}_m // G$ (the Hamiltonian reduction of the direct product of m Hamiltonian G -spaces).

Proposition 5.5. ([10]) $\widetilde{\mathcal{M}^*}(\mathbf{a})$ is isomorphic to the symplectic quotient

$$\widetilde{\mathcal{M}^*}(\mathbf{a}) \cong \tilde{O}_1 \times \cdots \times \tilde{O}_m // G$$

where $\tilde{O}_i \subset G \times \mathfrak{g}_{k_i}^*$ is the extended coadjoint orbit associated to $O_{A^0} \subset \mathfrak{b}_{k_i}^*$, the B_{k_i} coadjoint orbit containing the element ${}^i A^0 \in \mathfrak{b}_k^*$ (the chosen irregular type at a_i).

Quasi-Hamiltonian G -spaces and symplectic spaces of monodromy data. Let us recall the quasi-Hamiltonian description of the symplectic structure on the space of monodromy/Stokes data. Let $\theta, \bar{\theta}$ denote the left and right invariant \mathfrak{g} -valued Cartan one-forms on G respectively. Let ψ denote the canonical three-form of G , i.e., $\psi := \frac{1}{6} \langle \theta, [\theta, \theta] \rangle$.

Definition 5.6. ([3]) A complex manifold M is a complex quasi-Hamiltonian G -space if there is an action of G on M , a G -equivariant map $\mu : M \rightarrow G$ (where G acts on itself by conjugation), and a G -invariant holomorphic two-form $\omega \in \Omega^2(M)$ such that

- (i) $d\omega = \mu^*(\psi)$, where ψ is the canonical three-form on G ;
- (ii) $\omega(v_X, \cdot) = \frac{1}{2} \mu^*(\theta + \bar{\theta}, X) \in \Omega^1(M)$, for all $X \in \mathfrak{g}$, where v_X is the fundamental vector field $(v_X)_m = -\frac{d}{dt}(e^{tX} \cdot m)|_{t=0}$.

(iii) at each point $m \in M$, the kernel of ω is

$$\ker \omega_m = \{(v_X)_m \mid X \in \mathfrak{g} \text{ such that } hXh^{-1} = -X, \text{ where } h := \mu(m) \in G\}.$$

These axioms are motivated in terms of Hamiltonian loop group manifolds. See [3] for more details.

Example 5.7. Let $\mathcal{C} \subset G$ be a conjugacy class with the conjugation action of G and moment map μ given by the inclusion map. Then \mathcal{C} is a quasi-Hamiltonian G -space with two-form ω defined by

$$\omega_h(v_X, v_Y) = \frac{1}{2}(\langle X, \text{Ad}_h Y \rangle - \langle Y, \text{Ad}_h X \rangle),$$

for any $X, Y \in \mathfrak{g}$ and v_X, v_Y the fundamental vector field with respect to the conjugation action of G .

Similar to the Hamiltonian reduction, we have the following moment map reduction in the quasi-Hamiltonian setting.

Theorem 5.8. ([3]) Let (M, ω) be a quasi-Hamiltonian G -space with moment map $\mu : M \rightarrow G$. If the quotient $\mu^{-1}(1)/G$ of the inverse image $\mu^{-1}(1)$ of the identity is a manifold, then the restriction of ω to $\mu^{-1}(1)$ descends to a symplectic form on the reduced space $M//G := \mu^{-1}(1)/G$.

Definition 5.9. ([3]) Let M_1 (resp. M_2) be a quasi-Hamiltonian G -space with moment map μ_1 (resp. μ_2). Their fusion product $M_1 \otimes M_2$ is defined to be the quasi-Hamiltonian G -space $M_1 \times M_2$, where G acts diagonally, with two-form

$$\tilde{\omega} = \omega_1 + \omega_2 - \frac{1}{2}(\mu_1^* \theta, \mu_2^* \bar{\theta})$$

and moment map

$$\tilde{\mu} = \mu_1 \cdot \mu_2 : M \rightarrow G.$$

The quasi-Hamiltonian spaces from conjugacy classes can be seen as the building blocks of moduli spaces of flat connections on the trivial G -bundle on \mathbb{P}^1 . Indeed, following [3] let Σ_m be a sphere with m boundary components, the quasi-Hamiltonian reduction

$$\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_m // G$$

of the fusion product of m conjugacy classes \mathcal{C}_i is isomorphic to the moduli space of flat connections on Σ_m with the Atiyah-Bott symplectic form.

Let us next recall the building blocks of the monodromy data of meromorphic connections. Let T be a maximal torus of G with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ and B_{\pm} denote a pair of opposite Borel subgroups with $B_+ \cap B_- = T$. Let us consider the family of complex manifolds (see [12] for the geometrical origins of these spaces where their infinite-dimensional counterparts are described)

$$\tilde{\mathcal{C}} := \{(C, \mathbf{d}, \mathbf{e}, \lambda) \in G \times (B_- \times B_+)^{k-1} \times \mathfrak{t} \mid \delta(d_j)^{-1} = e^{\frac{\pi i \lambda}{k-1}} = \delta(e_j) \text{ for all } j\},$$

parameterised by an integer $k \geq 2$, where $\mathbf{b} = (d_1, \dots, d_{k-1})$, $\mathbf{e} = (e_1, \dots, e_{k-1})$ with $d_{\text{even}}, e_{\text{odd}} \in B_+$ and $d_{\text{odd}}, e_{\text{even}} \in B_-$ and $\delta : B_+ \rightarrow T$ is the homomorphism with kernel the unipotent subgroups U_{\pm} .

Proposition 5.10. ([12]) The manifold $\tilde{\mathcal{C}}$ is a complex quasi-Hamiltonian G -space with action

$$g \cdot (C, \mathbf{d}, \mathbf{e}, \lambda) = (Cg^{-1}, \mathbf{d}, \mathbf{e}, \lambda) \in \tilde{\mathcal{C}}, \quad \forall g \in G,$$

moment map

$$\mu : \tilde{\mathcal{C}} \rightarrow G; \quad (C, \mathbf{d}, \mathbf{e}, \lambda) \mapsto C^{-1}d_1^{-1} \cdots d_{k-1}^{-1}e_{k-1} \cdots e_1C,$$

and two-form

$$\omega = \frac{1}{2}(\bar{D}, \bar{E}) + \frac{1}{2} \sum_{j=1}^{k-1} (\mathcal{D}_j, \mathcal{D}_{j-1}) - (\mathcal{E}_j, \mathcal{E}_{j-1}).$$

Here $\bar{D} = D^*\bar{\theta}$, $\bar{E} = E^*\bar{\theta}$, $\mathcal{D}_j = D_j^*\theta$, $\mathcal{E}_j = E_j^*\theta \in \Omega^1(\tilde{\mathcal{C}}, \mathfrak{g})$ for maps $D_j, E_j : \tilde{\mathcal{C}} \rightarrow G$ defined by $D_i(C, \cdot, \cdot, \Lambda) = d_i \cdots d_1 C$, $E_i = e_i \cdots e_1 C$, $D := D_{k-1}$, $E := E_{k-1}$, $E_0 = D_0 := C$.

For instance, when $k = 2$, $\tilde{\mathcal{C}}_{k=2} \cong G \times G^*$ and the moment map, two form are given by

$$\mu = C^{-1}b_-^{-1}b_+C, \quad \omega = \frac{1}{2}(D^*\bar{\theta}, E^*\bar{\theta}) + \frac{1}{2}(D^*\theta, C^*\theta) - \frac{1}{2}(E^*\theta, C^*\theta) \quad (16)$$

where $D = b_-C$, $E = b_+C$.

For $k = 1$, we define $\tilde{\mathcal{C}}_{k=1} := \{(h, (e^{-\pi i \lambda}, e^{\pi i \lambda}, \lambda)) \mid h \in G, \lambda \in \mathfrak{t}'\}$ which is a submanifold of $\tilde{\mathcal{C}}_{k=2} \cong G \times G^*$, and thus inherits a G action. The restriction of the two form and moment map (16) of $\tilde{\mathcal{C}}_{k=2}$ to $\tilde{\mathcal{C}}_{k=1}$ makes it into a quasi-Hamiltonian G -space.

Given a divisor $D = \sum_{i=1}^m k_i(a_i)$ having each $k_i \geq 1$ at a_i on \mathbb{P}^1 . Let \mathcal{C}_i be the quasi-Hamiltonian G -space in Propostion 5.10 with $k = k_i$. Then the symplectic space $\widetilde{\mathcal{M}}(\mathbf{a})$ of monodromy data for compatibly meromorphic connections (V, ∇, \mathbf{g}) with irregular type \mathbf{a} can be described as follows.

Proposition 5.11 (Lemma 3.1 [12]). *The symplectic space $\widetilde{\mathcal{M}}(\mathbf{a})$ is isomorphic to the quasi-Hamiltonian quotient $\tilde{\mathcal{C}}_1 \otimes \cdots \otimes \tilde{\mathcal{C}}_m // G$, where \otimes denotes the fusion product of two quasi-Hamiltonian G -manifolds.*

The extension of the Atiyah-Bott symplectic structure to the case of singular \mathbb{C}^∞ -connections given in [10] leads to certain Hamiltonian loop group manifolds and $\tilde{\mathcal{C}}$ is the corresponding quasi-Hamiltonian space.

5.2 Irregular Riemann-Hilbert correspondence

Let \mathbf{a} be the data of a divisor $D = \sum k_i(a_i)$ and irregular types ${}^iA^0$ at each a_i . The irregular Riemann-Hilbert correspondence, which depends on a choice of tentacles τ (see Definition 3.9 in [10]), is a map ν from the global symplectic moduli space of meromorphic connections $\widetilde{\mathcal{M}}^*(\mathbf{a}) \cong (\tilde{\mathcal{O}}_1 \times \cdots \times \tilde{\mathcal{O}}_m) // G$ to the symplectic space of monodromy data $\widetilde{\mathcal{M}}(\mathbf{a}) \cong (\tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_m) // G$. In brief, the map arises as follows.

Let (P, ∇, \mathbf{g}) be a compatibly framed meromorphic connection on a holomorphic trivial G -bundle P with the irregular type \mathbf{a} . The chosen irregular type ${}^iA^0$ canonically determines some directions at a_i ('anti-Stokes directions'), and we can consider the Stokes sectors at each a_i bounded by these directions (and having some small fixed radius). Then the key fact is that, similar to the discussion in section 3, the framings \mathbf{g} (and a choice of branch of logarithm at each pole) determine,

in a canonical way, a choice of solutions of the equation $\nabla F = 0$ on each Stokes sector at each pole. Then along any path in the punctured sphere $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ between two Stokes sectors, we can extend the two corresponding canonical solutions and obtain an element in G by taking their ratio. The monodromy data of (P, ∇, \mathbf{g}) is simply the set of all such elements in G , plus the exponents of formal monodromy, thus corresponds to a point in the space of monodromy data $\tilde{\mathcal{C}}_1 \times \dots \times \tilde{\mathcal{C}}_m$. On the other hand, the triple (P, ∇, \mathbf{g}) represents a point in $\tilde{\mathcal{O}}_1 \times \dots \times \tilde{\mathcal{O}}_m$. Therefore, it produces a map from $\tilde{\mathcal{O}}_1 \times \dots \times \tilde{\mathcal{O}}_m$ to $\tilde{\mathcal{C}}_1 \times \dots \times \tilde{\mathcal{C}}_m$ by taking the monodromy data of meromorphic connections (P, ∇, \mathbf{g}) . Furthermore, this map is G -equivariant and descends to give the irregular Riemann-Hilbert map ν . The main result of [10] leads to:

Theorem 5.12. ([10]) The irregular Riemann-Hilbert map

$$\nu : (\tilde{\mathcal{O}}_1 \times \dots \times \tilde{\mathcal{O}}_m) // G \hookrightarrow (\tilde{\mathcal{C}}_1 \otimes \dots \otimes \tilde{\mathcal{C}}_m) // G \quad (17)$$

associating monodromy/Stokes data to a meromorphic connection on a trivial G -bundle P on \mathbb{P}^1 is a symplectic map (provided the symplectic structure on the right-hand side is divided by $2\pi i$).

We will analyze the case where the meromorphic connections have one pole of order one and one pole of order two, and show that the irregular Riemann-Hilbert map ν gives rise to a local symplectic isomorphism from (Σ, ω) to (Σ', ω') . Furthermore, given a choice of tangles, the corresponding map ν can be expressed explicitly by the connection matrix $C \in \text{Map}(\mathfrak{g}^*, G)$ defined in section 3. Thus with the help of Theorem 2.2, one can prove that C is a solution of the equation (5), i.e., $r_0^C = r_{\text{AM}}$.

First we need the following two propositions.

Proposition 5.13. *Let $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ be two copies of $\tilde{\mathcal{O}}$ with $k = 1$ and $k = 2$ respectively. Then the Hamiltonian quotient $\tilde{\mathcal{O}}_1 \times \tilde{\mathcal{O}}_2 // G$ is symplectic isomorphic to (Σ, ω) .*

Proof. By definition, $\tilde{\mathcal{O}}_1 = \{(g_1, x_1) \in G \times \mathfrak{g}^* \mid g_1 x_1 g_1^{-1} \in \mathfrak{t}'\}$ and $\tilde{\mathcal{O}}_2 = \{(g_2, A, x_2) \in G \times \mathfrak{g}^* \times \mathfrak{g}^* \mid \text{Ad}_{g_2} A = A_0\}$, where $A_0 \in \mathfrak{t}_{\text{reg}}$. Because A is determined by g_2 , $\tilde{\mathcal{O}}_2$ is naturally isomorphic to $G \times \mathfrak{g}^*$ by sending (g_2, A, x_2) to (g_2, x_2) . Note that the moment map is

$$\mu : \tilde{\mathcal{O}}_1 \times \tilde{\mathcal{O}}_2 \longrightarrow \mathfrak{g}^*; (g_1, x_1, g_2, x_2) \mapsto x_1 + x_2.$$

The submanifold $\mu^{-1}(0)$ is defined by $\mu^{-1}(0) := \{(g_1, x_1, g_2, -x_1) \in (G \times \mathfrak{g}^*)^2 \mid \text{Ad}_{g_1} x_1 \in \mathfrak{t}'\}$. We have a subjective map

$$\iota : \mu^{-1}(0) \longrightarrow \Sigma; (g_1, x_1, g_2, -x_1) \mapsto (g_2 g_1^{-1}, -\text{Ad}_{g_1}^* x_1)$$

whose fibres are the G orbits. Thus it induces an isomorphism from $\tilde{\mathcal{O}}_1 \times \tilde{\mathcal{O}}_2 // G$ to (Σ, ω) . To verify this is actually a symplectic isomorphism, let us take two tangents v_1, v_2 to $\mu^{-1}(0)$ which at each point $(g_1, x_1, g_2, -x_1)$ take the forms $v_i = (0, \text{Ad}_{g_1^{-1}} R_i, \text{Ad}_{g_2^{-1}} X_i, -\text{Ad}_{g_1^{-1}} R_i)$ for some $X_i \in \mathfrak{g}, R_i \in \mathfrak{t}^*$ and $i = 1, 2$ ($\mathfrak{g} \cong T_{g_2} G$ via left multiplication).

Let $\omega_{\mu^{-1}(0)}$ be the restriction of the (direct sum) symplectic structure $\omega_{\tilde{\mathcal{O}}_1 \times \tilde{\mathcal{O}}_2}$ on $\mu^{-1}(0)$. Following the formula (15), we have that at $(g_1, x_1, g_2, -x_1)$,

$$\begin{aligned} \omega_{\mu^{-1}(0)}(v_1, v_2) &= \omega_{\tilde{\mathcal{O}}_1}((0, \text{Ad}_{g_1^{-1}} R_1), (0, \text{Ad}_{g_1^{-1}} R_2)) \\ &\quad + \omega_{\tilde{\mathcal{O}}_2}((\text{Ad}_{g_2^{-1}} X_1, -\text{Ad}_{g_1^{-1}} R_1), (\text{Ad}_{g_2^{-1}} X_2, -\text{Ad}_{g_1^{-1}} R_2)) \\ &= \langle R_2, \text{Ad}_{g_1 g_2^{-1}} X_1 \rangle - \langle R_1, \text{Ad}_{g_1 g_2^{-1}} X_2 \rangle - \langle x_1, \text{Ad}_{g_2^{-1}}([X_1, X_2]) \rangle. \end{aligned}$$

On the other hand, a direct computation gives $\iota_*(v_i) = (\text{Ad}_{g_1 g_2^{-1}} X_i, -R_i)$ at $(g_2 g_1^{-1}, \text{Ad}_{g_1} x_1)$, here $\mathfrak{g} \cong T_{g_2 g_1^{-1}} G$ via left multiplication. Formula (6) makes it transparent that at $(g_2 g_1^{-1}, -\text{Ad}_{g_1}^* x_1) \in \Sigma$,

$$\begin{aligned} \omega(\iota_*(v_1), \iota_*(v_2)) &= \omega((\text{Ad}_{g_2 g_1^{-1}} X_1, -R_1), (\text{Ad}_{g_2 g_1^{-1}} X_2, -R_2)) \\ &= \langle R_2, \text{Ad}_{g_1 g_2^{-1}} X_1 \rangle - \langle R_1, \text{Ad}_{g_1 g_2^{-1}} X_2 \rangle - \langle x_1, \text{Ad}_{g_2^{-1}}([X_1, X_2]) \rangle. \end{aligned}$$

Therefore, we have that $\iota^* \omega = \omega_{\mu^{-1}(0)}$, i.e., ι induces a symplectic isomorphism between $\tilde{O}_1 \times \tilde{O}_2 // G$ and (Σ, ω) . ■

As for the Poisson Lie counterpart, we have

Proposition 5.14. *Let $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ be two copies of $\tilde{\mathcal{C}}$ with $k = 1$ and $k = 2$ respectively. Then the quasi-Hamiltonian reduction of the fusion of $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ is isomorphic as a symplectic manifold to the symplectic submanifold (Σ', ω') of the double Γ .*

Proof. We assume that the Borels chosen at the first pole are opposite to those chosen at the second (which we may since isomonodromy will give symplectic isomorphisms with the spaces arising from any other choice of Borels intersecting in T). Thus we have,

$$\tilde{\mathcal{C}}_1 = \{(h, e_-^{2\pi i \lambda}) \mid h \in G, \lambda \in \mathfrak{t}'\}, \quad \tilde{\mathcal{C}}_2 = \{(C, (b_-, b_+, \Lambda)) \mid \delta(b_{\pm}) = e^{\pm \pi i \Lambda}\},$$

here $e_-^{2\pi i \lambda} = (e^{\pi i \lambda^\vee}, e^{-\pi i \lambda^\vee}, \lambda^\vee)$ (exponential map of \mathfrak{g}^* with the opposite Borels chosen). The moment map on $\tilde{\mathcal{C}}_1 \otimes \tilde{\mathcal{C}}_2$ is $\mu = h^{-1} e^{-2\pi i \lambda^\vee} h C^{-1} b_-^{-1} b_+ C$. Therefore the condition $\mu = 1$ becomes $C e^{2\pi i \text{Ad}_{h^{-1}} \lambda^\vee} C^{-1} = b_-^{-1} b_+$, where $B := \text{Ad}_{h^{-1}}(\lambda^\vee)$. Recall that Σ' is a submanifold of Lu-Weinstein symplectic double Γ ,

$$\Sigma' := \{(g_1, e_-^{2\pi i \lambda}, g_2, (b_-, b_+, \Lambda)) \in \Gamma \mid \delta(b_{\pm}) = e^{\pm \pi i \Lambda}, g_1 e^{\pm \pi i \lambda^\vee} = b_{\pm} g_2\}.$$

We have a surjective map from $\mu^{-1}(1) = \{(h, e_-^{2\pi i \lambda}, C, (b_+, b_-, \Lambda)) \mid e^{2\pi i \text{Ad}_{h^{-1}} \lambda^\vee} = C b_-^{-1} b_+ C^{-1}\}$ to Σ' ,

$$(h, e_-^{2\pi i \lambda}, C, (b_+, b_-, \Lambda)) \rightarrow (Ch^{-1}, e^{-2\pi i \lambda}, u, (b_-, b_+, \Lambda))$$

whose fibres are precisely the G orbits, where $u := b_+^{-1} Ch^{-1} e^{\pi i \lambda} \in G$. Therefore, it induces an isomorphism from $\tilde{\mathcal{C}}_1 \otimes \tilde{\mathcal{C}}_2 // G$ to Σ' . An explicit formula for the symplectic structure on Σ' can be computed by using Theorem 3 of [2]. On the other hand we have an explicit formula for the symplectic structure on $\tilde{\mathcal{C}}_1 \otimes \tilde{\mathcal{C}}_2 // G$. A straightforward calculation shows these explicit formulae on each side agree. ■

To specify an irregular Riemann-Hilbert map, we have to make a choice of tentacles (see [10]). We introduce coordinate z to identify \mathbb{P}^1 with $\mathbb{C} \cup \infty$ and assume the divisor $D = 1(a_1) + 2(a_2)$ where $a_2 = 0$ and $a_1 = \infty$. Then we consider the meromorphic connections ∇ on a trivial holomorphic G -bundle P on \mathbb{P}^1 with compatible framings \mathbf{g} , such that (P, ∇, \mathbf{g}) have an irregular type $\frac{A_0}{z^2}$ at 0, where $A_0 \in \mathfrak{t}_{\text{reg}}$. Let us take a prior Stokes sector Sect_0 between two Stokes rays (only depend on A_0) at 0, and make a choice of tentacles as follows.

- (i) A choice of a point p_2 in Sect_0 at 0 and a point p_1 in Sect_0 near ∞ .

- (ii) A lift \hat{p}_i of each p_i to the universal cover of a punctured disc $D_i \setminus \{a_i\}$ containing p_i for $i = 1, 2$.
- (iii) A base point p_0 which coincides with p_1 .
- (iv) A contractible path $\gamma : [0, 1] \rightarrow \mathbb{P}^1 \setminus \{0, \infty\}$ in the punctured sphere, from p_0 to p_1 .

Note that the chosen point \hat{p}_2 determines a branch of $\log z$ on Sect_0 . According to section 3, let $C \in \text{Map}(\mathfrak{g}^*, G)$ be the connection matrix associated to $A_0 \in \mathfrak{t}_{\text{reg}}$, the choice of Sect_0 and the branch of $\log z$. Then we have

Proposition 5.15. *For the above choice of tentacles, the corresponding irregular Riemann-Hilbert map $\nu : (\tilde{O}_1 \times \tilde{O}_2) // G \cong \Sigma \rightarrow (\tilde{C}_1 \times \tilde{C}_2) // G \cong \Sigma'$ is given by*

$$\nu(h, \lambda) = (C(\text{Ad}_h^* \lambda)h, e^{2\pi i \lambda}, u, u^*), \quad \forall (h, \lambda) \in \Sigma,$$

for certain $u \in G, u^* \in G^*$ satisfying $C(\text{Ad}_h^* \lambda)h e^{2\pi i \lambda} = u^* u$.

Proof. Let $(P, \nabla, \mathbf{g} = (g_1, g_2))$ be a compatibly framed meromorphic connection with irregular type $\frac{A_0}{z^2}$ at a_2 , where $g_1, g_2 \in G$ and $A_0 \in \mathfrak{t}_{\text{reg}}$.

Upon trivializing V , we assume (P, ∇, \mathbf{g}) represents a point $(g_1, -x, g_2, A, x) \in \tilde{O}_1 \times \tilde{O}_2$, which means that in the trivialization, $\nabla = d - (\frac{A}{z^2} + \frac{x}{z})dz$. Furthermore, the given irregular type $\frac{A_0}{z^2}$ of ∇ in the compatible frame g_2 indicates that $\text{Ad}_{g_2} A = A_0$. Using the convention in Section 3.1, let F_0 (resp. F_∞) be the canonical solution of $\nabla_{A_0} F := dF - (\frac{A_0}{z^2} + \frac{\text{Ad}_{g_2} x}{z})Fdz = 0$ at 0 (resp. ∞). Due to the chosen frame \mathbf{g} , $\Phi_0 = g_2^{-1} F_\infty g_2$, $\Phi_1 = g_2^{-1} F_\infty g_2 g_1^{-1}$ and $\Phi_2 = g_2^{-1} F_0$ are the canonical solutions of $\nabla \Phi = 0$ on a neighbourhood of $p_0 = p_1$ and p_2 with respect to the compatible framing 1, g_1 and g_2 respectively. Then the monodromy data of (P, ∇, \mathbf{g})

$$(C_1, e_-^{2\pi i \lambda}, C_2, (b_-, b_+, \Lambda)) \in \tilde{C}_1 \times \tilde{C}_2 \cong G \times e_-^{\epsilon'} \times G \times G^*,$$

($e^{2\pi i \lambda} = (e^{\pi i \lambda^\vee}, e^{-\pi i \lambda^\vee}, \lambda^\vee)$ is the exponential map of \mathfrak{g}^* with the opposite Borels chosen) is the set of connection matrices C_i (the ratio of the canonical solutions Φ_i at p_i with Φ_0 at p_0 for $i = 1, 2$), as well as the Stokes data (b_-, b_+) at 0 and the formal monodromy at 0, ∞ . They are explicitly described as follows.

- along the path γ in the punctured sphere $\mathbb{P}^1 \setminus \{0, \infty\}$, we extend the two solutions Φ_0 and Φ_2 , then $\Phi_2 C_2 = \Phi_0$. Therefore we have $C_2 = F_0^{-1} F_\infty g_2$. By definition, $F_0^{-1} F_\infty = C(\text{Ad}_{g_2} x)$ the connection matrix of $\nabla_{A_0} F := dF - (\frac{A_0}{z^2} + \frac{\text{Ad}_{g_2} x}{z})Fdz = 0$;
- p_0, p_1 can be seen as connected by an identity path, thus $\Phi_1 C_1 = \Phi_0$. Therefore C_1 is equal to g_1 , the ratio of the frame chosen at p_0 and p_1 ;
- b_-, b_+ at 0 are the Stokes matrices of ∇_{A_0} , which are determined by the monodromy relation (3.5).

Therefore, the chosen tentacle determines a map $\nu' : \mu^{-1}(0) \subset \tilde{O}_1 \times \tilde{O}_2 \rightarrow \mu'^{-1}(0) \subset \tilde{C}_1 \times \tilde{C}_2$, which is given by

$$\nu'(g_1, -x, g_2, x) = (g_1, e_-^{-2\pi i \text{Ad}_{g_1} x}, C(\text{Ad}_{g_2} x)g_2, (b_-, b_+, \Lambda)).$$

This map ν' is G -equivariant and descends to the irregular Riemann-Hilbert map $\nu : \Sigma \rightarrow \Sigma'$

$$\nu(h, \lambda) = (C(\text{Ad}_h^* \lambda)h, e^{2\pi i \lambda}, u, u^*),$$

for any $(h, \lambda) \in \Sigma$. Here $u \in G, u^* \in G^*$ satisfy $C(\text{Ad}_h^* \lambda) h e^{2\pi i \lambda} = u^* u$, and we use the isomorphisms $\mu^{-1}(0)/G \cong \Sigma$ and $\mu'^{-1}(1)/G \cong \Sigma'$ constructed in Proposition 5.13 and 5.14 respectively. ■

Now we can give a proof of our main theorem:

The proof of Theorem 3.4. Following Proposition 5.15, the irregular Riemann-Hilbert map ν coincides with the local diffeomorphic map $F'_C : \Sigma \rightarrow \Sigma'$ defined in Section 2. According to Corollary 2.3, $\nu : (\Sigma, \omega) \rightarrow (\Sigma', \omega')$ is a symplectic map, provided the symplectic structure on the right-hand side is divided by $2\pi i$, if and only if $C_{2\pi i}$ is a solution of the equation (5) (Here $C_{2\pi i}(x) = C(\frac{1}{2\pi i}x)$, for all $x \in \mathfrak{g}^*$). However, the former is guaranteed by Theorem 5.12. As a result, we get a proof of Theorem 3.4, i.e., $C_{2\pi i}$ is a solution of the gauge transformation equation (5). ■

6 Lu-Weinstein symplectic groupoids via Alekseev-Meinrenken r -matrices

As an application of the above construction, we describe the Lu-Weinstein symplectic groupoids via Alekseev-Meinrenken r -matrices.

Following the construction of section 2.1 in [38], for any r -matrix with the skew-symmetric part r_0 ,

$$\pi_{AM}(h, x) := \pi_{KKS}(x) + l_h(\theta) + l_h(r_{AM}(x)) - r_h(r_0),$$

defines a symplectic structure on $G \times \mathfrak{g}^*$, where $\theta := \frac{\partial}{\partial x^a} \wedge e_a \in \Gamma(\wedge^2(T\mathfrak{g}^* \oplus \mathfrak{g}))$ for a base $\{e_a\}$ of \mathfrak{g} and the corresponding coordinates $\{x^a\}$ on \mathfrak{g}^* . We will prove that $(G \times \mathfrak{g}^*, \pi_{AM})$ is a natural symplectic groupoid and is locally symplectic isomorphic to Lu-Weinstein double symplectic groupoid with respect to (\mathfrak{g}, r) .

To do this, let us consider the Semenov-Tian-Shansky (STS) Poisson tensor on \mathfrak{g}^* defined by

$$\pi_{STS}(x)(df, dg) = \langle df(x) \otimes dg(x), \text{ad}_x \otimes \frac{1}{2} \text{ad}_x \coth(\frac{1}{2} \text{ad}_x)(t) - \otimes^2 \text{ad}_x(r_0) \rangle,$$

for any $f, g \in C^\infty(\mathfrak{g}^*)$. We denote by L, R the group morphisms corresponding to the Lie algebra morphisms $L, R : \mathfrak{g}^* \rightarrow \mathfrak{g}$

$$L(x) := (x \otimes \text{id})(r), \quad R(x) := -(x \otimes \text{id})(r^{2,1}) \quad \forall x \in \mathfrak{g}^*.$$

Let (G^*, π_{G^*}) be the simply connected Poisson Lie group associated to the quasitriangular Lie bialgebra (\mathfrak{g}, r) .

Proposition 6.1. [23] The map $I : (\mathfrak{g}^*, \pi_{STS}) \rightarrow (G^*, \pi_{G^*})$ determined by $e^{x^\vee} = L(I(x))^{-1} R(I(x))$ for any $x \in \mathfrak{g}^*$, is a Poisson map.

Actually, the STS Poisson structure on \mathfrak{g}^* is completely determined by the above proposition. Now let us take any solution $g \in \text{Map}(\mathfrak{g}^*, G)$ of the gauge transformation equation (5).

Theorem 6.2. $(G \times \mathfrak{g}^*, \pi_{AM})$ is a Poisson (therefore symplectic) groupoid over \mathfrak{g}^* with the structure

maps given by

$$\begin{aligned} \alpha(h, x) &\rightarrow x \in \mathfrak{g}^*, \quad \beta(h, x) \rightarrow \text{Ad}_h^* x \in \mathfrak{g}^*, \quad \forall h \in G, x \in \mathfrak{g}^*, \\ \varepsilon : \mathfrak{g}^* &\rightarrow G \times \mathfrak{g}^* : x \mapsto (g(x), \text{Ad}_{g(x)}^* x) \in G \times \mathfrak{g}^*, \\ m : \mathcal{G}_2 &:= \{((h_1, x), (h_2, y)) \in \mathcal{G} \times \mathcal{G} \mid \beta(h_1) = \alpha(h_2)\} \rightarrow \mathcal{G} : \\ &(h_1, x), (h_2, y = \text{Ad}_{h_1}^* x) \mapsto \text{Ad}_{h_1 h_2}^* x, \quad \forall (h_1, h_2) \in \mathcal{G}_2. \end{aligned}$$

Furthermore, it is the integration of the STS Poisson structure on \mathfrak{g}^* .

A straightforward proof can be obtained verifying the following equivalent conditions one by one.

Lemma 6.3. ([37]) $(\mathcal{G} \Rightarrow M, \pi)$ is a Poisson groupoid if and only if all the following hold.

(1) For all $(x, y) \in \mathcal{G}_2$,

$$\pi(xy) = R_Y \pi(x) + L_X \pi(y) - R_Y L_X \pi(m),$$

where $m = \beta(x) = \alpha(y)$ and X, Y are (local) bisections through x and y respectively.

(2) M is a coisotropic submanifold of \mathcal{G} ,

(3) For all $x \in \mathcal{G}$, $\alpha_* \pi(x)$ and $\beta_* \pi(x)$ only depend on the base points $\alpha(x)$ and $\beta(y)$ respectively.

(4) For all $f, g \in C^\infty(M)$, one has $\{\alpha^* f, \beta^* g\} = 0$,

(5) The vector field $X_{\beta^* f}$ is left invariant for all $f \in C^\infty(M)$.

Theorem 6.4. The map $v : G \times \mathfrak{g}^* \rightarrow G \times G^*$, $v(h, x) = (g(x)h, g^*(x))$ for all $(h, x) \in G \times \mathfrak{g}^*$, gives a local symplectic isomorphism from $(G \times \mathfrak{g}^*, \pi_{AM})$ to Lu-Weinstein symplectic double Γ , where $g^* : \mathfrak{g}^* \rightarrow G^*$ is the unique local isomorphism defined by the identity

$$g(x)e^{x^\vee} g(x)^{-1} = L(g^*(x))R(g^*(x))^{-1}.$$

Actually, under the transformation $F : G \times \mathfrak{g}^* \rightarrow G \times \mathfrak{g}^*$ given by $F(h, x) = (hg^{-1}(x), \text{Ad}_{g(x)}^* x)$, then the Poisson tensor π_{AM} becomes

$$F_*(\pi_{AM})(h, x) = \pi_{STS}(x) + l_h(\theta^g(x)) + l_h(r_0) - r_h(r_0), \quad \forall (h, x) \in G \times \mathfrak{g}^*$$

where $\theta^g \in \Gamma(\wedge^2(T\mathfrak{g}^* \oplus \mathfrak{g}))$ is the gauge transformation of θ under $g \in \text{Map}(\mathfrak{g}^*, G)$ (see [38] for more details about a generalized dynamical r -matrix and its gauge transformations). According to [38], we have that $(\pi_{STS}, \theta^g, r_0)$ is a gauge transformation of the dynamical r -matrix $(\pi_{KKS}, \theta, r_{AM})$ under $g \in \text{Map}(\mathfrak{g}^*, G)$, thus also a generalized classical dynamical r -matrix. It therefore verifies that equation (5) is indeed a gauge transformation equation in the theory of generalized dynamical r -matrices.

A Appendix: Proof of Theorem 2.2

In this section, we will study in details the symplectic submanifold Σ' of Lu-Weinstein symplectic double groupoid Γ and then give a proof of Theorem 2.2. According to Section 2, given a quasitriangular Lie bialgebra (\mathfrak{g}, r) , (Γ, π_Γ) is the set

$$\Gamma = \{(h, h^*, u, u^*) \mid h, u \in G, h^*, u^* \in G^*, hh^* = u^*u\}$$

with the unique Poisson structure π_Γ such that the local diffeomorphism $(\Gamma, \pi_\Gamma) \rightarrow (D, \pi_D): (h, h^*, u, u^*) \mapsto hh^*$ is Poisson (D is the double of (\mathfrak{g}, r)). Then the submanifold Σ' takes the form

$$\Sigma' = \{(h, h^*, u, u^*) \in \Gamma \mid h^* \in e^{\mathfrak{t}'} \subset G^*\}.$$

Proposition A.1. Σ' is a symplectic submanifold of the Lu-Weinstein symplectic double (Γ, π_Γ) .

Proof. An explicit formula for the restriction of symplectic 2-form on $\Sigma' \in \Gamma$ can be computed by using Theorem 3 of [4]. One checks directly that it is symplectic. ■

Thus Σ' inherits a symplectic structure. We denote by π' the corresponding Poisson tensor. Note that the inclusion map $(\Sigma', \pi') \hookrightarrow (\Gamma, \pi_\Gamma)$ and the dressing transformation map $(\Gamma, \pi_\Gamma) \rightarrow (G^*, \pi_{G^*}); (h, h^*, u, u^*) \mapsto d_h(h^*)$ are Poisson, so is their composition. Thus we have

Proposition A.2. The map

$$P' : (\Sigma', \pi') \rightarrow (G^*, \pi_{G^*}); (h, e^\lambda, u, u^*) \mapsto d_h e^\lambda$$

is a Poisson map.

To simplify the notation, we take a local model of (Σ', π') as follows. Note that we have the local diffeomorphism

$$\Sigma' \rightarrow G \times e^{\mathfrak{t}'}; (h, e^\lambda, u, u^*) \mapsto (h, e^\lambda).$$

To simplify the notation, we will take $G \times e^{\mathfrak{t}'}$ as a local model of (Σ', π) with the induced Poisson tensor, denoted also by π' . Generally, π' is only defined on a dense subset of $G \times e^{\mathfrak{t}'}$, however this is enough for our purpose.

Let T act on $G \times e^{\mathfrak{t}'}$ by $t \cdot (h, e^\lambda) = (ht, e^\lambda)$. The fibres of the map $P' : G \times \mathfrak{t}' \rightarrow G^*, (h, e^\lambda) \mapsto d_h(e^\lambda)$ are precisely the T -orbits. Thus a general 1-form on $G \times e^{\mathfrak{t}'}$ takes the form $P'^*(\beta) + \hat{\eta}$, where $\beta \in \Omega^1(G^*)$, $\eta \in \mathfrak{t}^* \subset \mathfrak{g}^*$ (via inner product) and at each point (h, e^λ) , $\hat{\eta} := (l_{h^{-1}} \circ r_{e^{-\lambda}})^* \eta$.

Proposition A.3. At each point (h, e^λ) , π' is given for any forms $\phi_1 := P'^*(\beta_1) + \hat{\eta}_1$, $\phi_2 := P'^*(\beta_2) + \hat{\eta}_2$ by

$$\begin{aligned} \pi'(h, e^\lambda)(\phi_1, \phi_2) &= \pi_{G^*}(d_h e^\lambda)(\beta_1, \beta_2) + \langle X_1, \eta_2 \rangle - \langle X_2, \eta_1 \rangle \\ &+ (l_{h^{-1}} \pi_G(h))(\eta_1, \xi_2) - (l_{h^{-1}} \pi_G(h))(\eta_2, \xi_1) + (l_{h^{-1}} \pi_G(h))(\eta_1, \eta_2), \end{aligned} \quad (18)$$

where $\xi_i + X_i \in \mathfrak{g}^* \otimes \mathfrak{g}$ is the pull back of $P'^*(\beta_i)$ under $l_{h^{-1}} \circ r_{e^{-\lambda}}$ for $i = 1, 2$.

Proof. Following [29], if $m \in D$ (the double Lie group) can be factored as $m = hu$ for some $h \in G$ and $u \in G^*$ (locally it is always the case), then explicit formula for π_D is given by

$$\begin{aligned} &((l_{h^{-1}} \circ r_{u^{-1}})(\pi_D))(m)(\xi_1 + X_1, \xi_2 + X_2) \\ &= \langle X_1, \xi_2 \rangle - \langle X_2, \xi_1 \rangle + (l_{h^{-1}} \pi_G(h))(\xi_1, \xi_2) + (r_{u^{-1}} \pi_{G^*}(u))(X_1, X_2) \end{aligned}$$

for $\xi_1 + X_1, \xi_2 + X_2 \in \mathfrak{g}^* \oplus \mathfrak{g}$.

On one hand, Proposition A.2 gives that

$$\pi'(P'^*(\beta_1), P'^*(\beta_2)) = \pi_{G^*}(\beta_1, \beta_2),$$

for any $\beta_1, \beta_2 \in \Omega^1(G^*)$.

On the other hand, let us consider the one form taking the form of $\hat{\eta} := l_{h^{-1}}^*(r_{e^{-\lambda}}^* \eta)$, $\eta \in \mathfrak{t}^* \subset \mathfrak{g}^*$. From the expression of π_D , we see that $\pi_D^\sharp(h e^\lambda)(\hat{\eta})$ is tangent to $G \times e^{\mathfrak{t}'}$ at (h, e^λ) . Thus

$$\begin{aligned} \pi'(\hat{\eta}_1 + P'^*(\beta_1), \hat{\eta}_2) &= \pi_D(\hat{\eta}_1 + d^*(\beta_1), \hat{\eta}_2)|_{G \times e^{\mathfrak{t}'}} \\ &= \langle X_1, \eta_2 \rangle + l_{h^{-1}} \pi_G(\xi_1, \eta_2) + l_{h^{-1}} \pi_G(\eta_1, \eta_2) \end{aligned}$$

where $\xi_1 + X_1 \in \mathfrak{g}^* \otimes \mathfrak{g}$ is the pull back of $P'^*(\beta_1)$ under $l_{h^{-1}} \circ r_{e^{-\lambda}}$. The above two identities indicate the expression (18) of π' . ■

In the following, we will give a description of the Poisson space $(G \times e^{\mathfrak{t}'}, \pi')$ by using r -matrices. Let us define a bivector field on $G \times \mathfrak{t}'$ which at each point $(h, \lambda) \in G \times \mathfrak{t}'$ takes the form

$$\pi_r(h, \lambda) = l_h(t_i) \wedge \frac{\partial}{\partial t^i} + l_h((\text{id} \otimes \text{ad}_{\lambda^\vee}^{-1})(t)) + l_h(r_{\text{AM}}(\lambda)) - r_h(r_0)$$

where $t \in S^2(\mathfrak{g})^\mathfrak{g}$ is the Casimir element, $\{t_i\}$ is a basis of \mathfrak{t} and $\{t^i\}$ the corresponding coordinates on \mathfrak{t}^* and at any point $x \in \mathfrak{g}$, $\text{ad}_x^{-1} : \mathfrak{g} \rightarrow \mathfrak{g}$ is the trivial extension of the map $\text{ad}_x^{-1} : \mathfrak{g}_x^\perp \rightarrow \mathfrak{g}_x^\perp \subset \mathfrak{g}$ corresponding to the decomposition $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}_x^\perp$. Here \mathfrak{g}_x is the isotropic subalgebra of \mathfrak{g} at x and \mathfrak{g}_x^\perp its complement with respect to the inner product. By using the dynamical Yang-Baxter equation of r_{AM} , one can show that π_r is a Poisson tensor.

Proposition A.4. *The image of π_r under the diffeomorphism $\Psi : G \times \mathfrak{t}' \rightarrow G \times e^{\mathfrak{t}'}$, $(h, \lambda) \mapsto (h, e^\lambda)$ coincides with π' .*

Proof. Recall that (from the discussion above Proposition A.3) a general 1-form on $G \times e^{\mathfrak{t}'}$ takes the form $P'^*(\beta) + \hat{\eta}$, where $\beta \in \Omega^1(G^*)$ and $\eta \in \mathfrak{t}^* \subset \mathfrak{g}^*$. We will prove that $\Psi_*(\pi_r)(\hat{\eta}, \cdot) = \pi'(\hat{\eta}, \cdot)$ and $\Psi_*(\pi_r)(P'^*(\beta), \cdot) = \pi'(P'^*(\beta), \cdot)$ respectively. First note that at each point (h, λ) (by the equivariance of r_{AM})

$$l_h(r_{\text{AM}}(\lambda) + (\text{id} \otimes \text{ad}_{\lambda^\vee}^{-1})(t)) = r_h(r_{\text{AM}}(x) + (\text{id} \otimes \text{ad}_{x^\vee}^{-1})(t)) = r_h((\text{id} \otimes \coth(\frac{1}{2}\text{ad}_{x^\vee}))(t)),$$

where $x = \text{Ad}_h^* \lambda \in \mathfrak{g}^*$. By the definition of the map P , a direct calculation gives that

$$\pi_r(h, \lambda)(P^*(\alpha_1), P^*(\alpha_2)) = (\text{ad}_{x^\vee}^* \otimes \frac{1}{2}\text{ad}_{x^\vee}^* \coth(\frac{1}{2}\text{ad}_{x^\vee}^*)(t) - \otimes^2 \text{ad}_{x^\vee}^*(r_0))(\alpha_1, \alpha_2),$$

where $(h, \lambda) \in G \times \mathfrak{t}'$ and $x = \text{Ad}_h^* \lambda$. In other words,

$$\pi_r(h, \lambda)(P^*(\alpha_1), P^*(\alpha_2)) = \pi_{\text{STS}}(x)(\alpha_1, \alpha_2).$$

On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} G \times \mathfrak{t}' & \xrightarrow{\Psi} & G \times e^{\mathfrak{t}'} \\ P \downarrow & & P' \downarrow \\ \mathfrak{g}^* & \xrightarrow{I} & G^* \end{array},$$

where $I : (\mathfrak{g}^*, \pi_{\text{STS}}) \rightarrow (G^*, \pi_{G^*})$ is the local Poisson isomorphism defined in Section 6. Thus $P^*(I^*(\beta_i)) = \Psi^*(P'^*(\beta_i))$ for any $\beta_i \in \Omega^1(G^*)$, $i = 1, 2$. Therefore,

$$\begin{aligned} \Psi_* \pi_r(P'^*(\beta_1), P'^*(\beta_2)) &= \pi_r(P^*(I^*(\beta_1)), P^*(I^*(\beta_2))) = \pi_{\text{STS}}(I^*(\beta_1), I^*(\beta_2)), \\ \pi'(P'^*(\beta_1), P'^*(\beta_2)) &= \pi_{G^*}(\beta_1, \beta_2). \end{aligned}$$

Combining with Proposition 6.1 which says $\pi_r(I^*(\beta_1), I^*(\beta_2)) = \pi_{G^*}(\beta_1, \beta_2)$, we have that

$$\Psi_*\pi_r(P'^*(\beta_1), P'^*(\beta_2)) = \pi'(P'^*(\beta_1), P'^*(\beta_2)).$$

For the remaining part, by the definition of the diffeomorphism Ψ and the expression of $\pi_G = l_h(r_0) - r_h(r_0)$, one can easily get that

$$\Psi_*(\pi_r)(P'^*(\beta_1) + \hat{\eta}_1, \hat{\eta}) = \langle X_1, \eta \rangle + l_{h^{-1}}\pi_G(\xi_1, \eta) + l_{h^{-1}}\pi_G(\eta_1, \eta),$$

where $\xi_1 + X_1 \in \mathfrak{g}^* \otimes \mathfrak{g}$ is the pull back of $P'^*(\beta_1)$ under $l_{h^{-1}} \circ r_{e^{-\lambda}}$ and $\eta, \eta_1 \in \mathfrak{t}'$. By comparing with the expression of π' , we have that $\Psi_*(\pi_r)(\hat{\eta}, \cdot) = \pi'(\hat{\eta}, \cdot)$ for any $\eta \in \mathfrak{t}^*$.

Eventually, we prove that $\Psi_*(\pi_r)(P'^*(\beta) + \hat{\eta}, \cdot) = \pi'(P'^*(\beta) + \hat{\eta}, \cdot)$ for any $\beta \in \Omega^1(G^*)$, $\eta \in \mathfrak{t}^* \subset \mathfrak{g}^*$. That is, the image of π_r under the diffeomorphism Ψ coincides with π' . ■

In other words, we have a local symplectic isomorphism (denoted by same symbol) $\Psi : (G \times \mathfrak{t}', \pi_r) \rightarrow (\Sigma', (h, \lambda) \mapsto (h, e^\lambda, u, u^*))$, where $u \in G, u^* \in G^*$ are unique determined by the identity $he^\lambda = u^*u$.

For the Poisson tensor π corresponding to the symplectic form ω on $G \times \mathfrak{t}'$, we have

Proposition A.5. *The Poisson tensor π takes the form*

$$\pi(h, \lambda) = l_h(t_j) \wedge \frac{\partial}{\partial t^j} + l_h(\text{id} \otimes (\text{ad}_{\lambda^\vee}^{-1})(t))$$

where $\{t_j\}$ is a basis of \mathfrak{t} and $\{t^j\}$ the corresponding coordinates on \mathfrak{t}^* .

After these preliminary work, we can give a proof of Theorem 2.2. Following Proposition A.4, $(G \times \mathfrak{t}', \pi_r)$ is locally isomorphic to (Σ', π') . Therefore we can take $(G \times \mathfrak{t}', \pi_r)$ as a local model of (Σ', π') and then the map defined by (7) becomes $F_g : \Sigma = G \times \mathfrak{t}' \rightarrow G \times \mathfrak{t}'$, $(h, \lambda) \mapsto (g(x)h, \lambda)$, where $x = \text{Ad}_h^*\lambda$. Theorem 2.2 is thus equivalent to that

Theorem A.6. *$F_g : (G \times \mathfrak{t}', \pi) \rightarrow (G \times \mathfrak{t}', \pi_r)$ is a Poisson map if and only if $g \in \text{Map}(\mathfrak{g}^*, G)$ satisfies the gauge transformation equation (5), $r_0^g = r_{\text{AM}}$.*

Proof. We only need to show that $F_{g*}\pi = \pi_r$ is equivalent to the equation $r_0^g = r_{\text{AM}}$. By comparing the expressions of π and π_r , we have

$$\pi_r(h, \lambda) = \pi(h, \lambda) + l_h(r_{\text{AM}}(\lambda) - \otimes^2 \text{Ad}_{h^{-1}}(r_0)).$$

At any point $(h, \lambda) \in G \times \mathfrak{t}'$, $F_g(h, \lambda) = (g(x)h, \lambda)$ where $x := \text{Ad}_h^*\lambda \in \mathfrak{g}^*$. We take $\{e^i\}, \{e_i\}$ as dual bases of $\mathfrak{g}^*, \mathfrak{g}$ and $\{t^j\}, \{t_j\}$ dual bases of \mathfrak{t}^* and \mathfrak{t} . A straightforward calculation gives that at each point $(g(x)h, \lambda) \in G \times \mathfrak{t}'_{\text{reg}}$

$$\begin{aligned} F_{g*}(l_h(e_i)) &= l_{gh}(e_i) + l_{gh}(h^{-1}g^{-1}\frac{\partial g}{\partial X^i}h), \\ F_{g*}(\frac{\partial}{\partial t_j}) &= \frac{\partial}{\partial t_j} + l_{gh}(h^{-1}g^{-1}\frac{\partial g}{\partial T^j}h) \end{aligned}$$

where $X^i := [\text{Ad}_h e_i, x]$, $T^j := \text{Ad}_h^* t^j$ are tangent vectors at $x = \text{Ad}_h^*\lambda$. Note that $T^m \in \mathfrak{g}_x$ (the isotropic subalgebra at x) and X^i span the tangent space $T_x \mathfrak{g}^*$ and thus the above formulas

involve all the possible derivative of $g \in \text{Map}(\mathfrak{g}^*, G)$. A direct computation shows that at each point $(g(x)h, \lambda) \in G \times \mathfrak{t}'$ (here $x = \text{Ad}_h^* \lambda \in \mathfrak{g}^*$)

$$F_{g*}(\pi)(g(x)h, \lambda) = \pi(g(x)h, \lambda) + l_{gh}(\otimes^2 \text{Ad}_{h^{-1}} U(x)),$$

where $U(x) \in \mathfrak{g} \wedge \mathfrak{g}$ is defined, by using the notation in Theorem 2.1, as

$$U(x) = g_1^{-1} d_2(g_1) - g_2^{-1} d_1(g_2) + \langle \text{id} \otimes \text{id} \otimes x, [g_1^{-1} d_3(g_1), g_2^{-1} d_3(g_2)] \rangle.$$

Thus by comparing with the expression of π_r ,

$$\pi_r(g(x)h, \lambda) = \pi(g(x)h, \lambda) + l_{gh}(r_{AM}(\lambda) - \otimes^2 \text{Ad}_{(gh)^{-1}} r_0),$$

we obtain that $F_{g*}(\pi) = \pi_r$ at point $(g(x)h, \lambda) \in G \times \mathfrak{t}'$ if and only if

$$r_{AM}(\lambda) = \otimes^2 \text{Ad}_{(gh)^{-1}} r_0 + \otimes^2 \text{Ad}_{h^{-1}} U(x).$$

Note that $x = \text{Ad}_h^* \lambda$, by the equivariance of r_{AM} , we have $\otimes^2 \text{Ad}_h r_{AM}(\lambda) = r_{AM}(x)$. Thus the above formula is exactly the gauge transformation equation $r_0^g = r_{AM}$. ■

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